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THE *K*-THEORY OF STRICT HENSEL LOCAL RINGS AND A THEOREM OF SUSLIN

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Introduction

In his remarkable article [11] Suslin proved a conjecture of Quillen and Lichtenbaum for fields of positive characteristic:

Theorem. *Let F be an algebraically closed field of characteristic* $p>0$. *Then* $K_i(F)$ *is divisible if* $i>1$, *with zero torsion if i is even, while if* $i=2j-1\geq 1$ is odd:

$$K_{2j-1}(F)_{\mathrm{tors}}\simeq \varinjlim_{(n,p)=1} \mu_n(F)^{\otimes j}.$$

In fact Suslin proved, more generally, that if $F_0\subset F$ is an extension of algebraically closed fields, then the natural map $K_*(F_0;\mathbb{Z}/n)\to K_*/;\mathbb{Z}/n)$ is an isomorphism, thereby reducing the theorem above to a result of Quillen.

In this note we show how Suslin's result may be extended to extensions of strict local Hensel rings:

Theorem A. *Let R be the strict Henselization of the local ring at a smooth point of a variety of finite type over a separably closed field k. Then for n prime to* char(*k*)*:*

$$K_*(k;\mathbb{Z}/n)\xrightarrow{\sim}K_*(R;\mathbb{Z}/n).$$

Using this result, we are able to obtain partial confirmation of the conjecture in characteristic zero:

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Theorem B. *Let k be an algebraically closed field of characteristic zero (e.g. $\mathbb{C}$) and l any prime. Then:*

(i) *For all $i \geq 0$ and $v \geq 1$, $K_i(k, \mathbb{Z}/l^v)$ is finite.*

(ii) *If $0 \leq i \leq 2l-4$:*

$$K_i(k, \mathbb{Z}/l^v) \cong \begin{cases} \mathbb{Z}/l^v, & i \text{ even}, \\ 0, & i \text{ odd}. \end{cases}$$

(iii) *For all $i \geq 0$,*

$$\left(\varprojlim_{v} K_i(k, \mathbb{Z}/l^v)\right) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}_l, & i \text{ even}, \\ 0, & i \text{ odd}. \end{cases}$$

Our method is first of all to modify the proof of Suslin's rigidity theorem [11] by constructing a pairing, for X a smooth connected curve over a field k with $1/n \in k$, and n odd or divisible by 4:

$$H_c^2(X; \mu_n)_{\text{alg}} \otimes K_q(X; \mathbb{Z}/n) \to K_q(k; \mathbb{Z}/n)$$

where $H_c^2(X; \mu_n)_{\text{alg}}$ is the subgroup of $H_c^2(X; \mu_n)$ generated by classes of divisors on X. In Sections 3 and 4 we use this pairing to recover Suslin's main theorem and its generalization to strict Hensel local rings (Theorem A). Finally in Section 5, using techniques from [12], we prove Theorem B.

After the first version of this paper was written, we learned that Gabber has independently obtained the results of Sections 1-4. Also Suslin, using these results, has proved the Quillen-Lichtenbaum conjecture for algebraically closed fields of characteristic zero. More recently, Jardine has given a simple, elegant proof of this conjecture for all algebraically closed fields, again using Theorem A.

In several places we have made assertions about $K_*(, \mathbb{Z}/n)$ and justified them by references to the original paper of Quillen [9] even though K-theory with coefficients does not appear there. It is however an easy exercise to see that the theorems we refer to are equally true for ordinary K-theory and for K-theory with coefficients. This is easily seen by noting that the proofs in [9] show that certain maps of K-theory spectra are weak homotopy equivalences, or are nullhomotopic, or fit into homotopy fibre sequences. Applying the homotopy group functor $\pi_*( )$ to these gives the standard results, and applying $\pi_*(; \mathbb{Z}/n)$ gives the analogous results for $K_*(; \mathbb{Z}/n)$.

We note that $K_*(; \mathbb{Z}/n)$ is annihilated by n if n is odd or divisible by 4, and is annihilated by $2n$ in general, see [2, 1.5] or [8, 7.1]. (As we are dealing with spectra and not ordinary spaces, we do not have to make exceptions for homotopy in low dimensions as in [2] or [8].)

Finally, we assume that all schemes appearing are separated.

1. Generalized Jacobians and cohomology with compact supports

Let X be a nonsingular connected curve over a field k, $\bar{X}$ its nonsingular projective model. Write $S=\bar{X}-X$, viewed as a reduced subscheme of $\bar{X}$, and $j: X\to\bar{X}$, $i: S\to\bar{X}$ for the inclusions.

A divisor D on X is as usual a finite formal sum $\sum_{P\in X} n_P[P]$, so $\mathrm{Div}(X)=\bigoplus_{x\in X}\mathbb{Z}$, the sum being over closed points. Let $\mathcal{M}=\sum_{P\in S}[P]$, viewed as a 'module' in the sense of Serre [10]. If $D\in\mathrm{Div}(X)$, recall that $D\sim_{\mathcal{M}}0$ if there is a rational function $f\in k(X)$ with $f\equiv 1\ (\mathcal{M})$ such that $\mathrm{div}(f)=D$. Then we set

$$\mathrm{Pic}(\bar{X},S)=\mathrm{Div}(X)/\{D\mid D\sim_{\mathcal{M}}0\}.$$

If k is algebraically closed, then the subgroup of $\mathrm{Pic}(\bar{X},S)$ consisting of divisors of degree zero is isomorphic to the group of k-rational points of the generalized Jacobian $J_{\mathcal{M}}(\bar{X})$.

Let $\mathbb{H}^*_{\mathrm{et}}(\bar{X};\mathbb{G}_m\to i_*\mathbb{G}_m)$ be the hypercohomology of $\bar{X}$ with coefficients in the complex of sheaves $\mathbb{G}_m\to i_*\mathbb{G}_m$ with $\mathbb{G}_m$ in degree zero.

Lemma 1.1. $\mathrm{Pic}(\bar{X},S)\simeq\mathbb{H}^1_{\mathrm{et}}(\bar{X};\mathbb{G}_m\to i_*\mathbb{G}_m)$.

Proof. By descent $H^i_{\mathrm{et}}(Z,\mathbb{G}_m)=H^i_{\mathrm{Zar}}(Z,\mathcal{O}^*_Z)$ for $i=0,1$ and $Z=\bar{X}$ or S, so it is enough to show that

$$\mathrm{Pic}(\bar{X},S)\simeq\mathbb{H}^1_{\mathrm{Zar}}(\bar{X};\mathcal{O}^*_{\bar{X}}\to i_*\mathcal{O}^*_S).$$

Consider the diagram of sheaves on $\bar{X}$ (where $u:\mathrm{Spec}(k(X))\to\bar{X}$, $v_x:\{x\}\to\bar{X}$, $w:\mathrm{Spec}(\mathcal{O}_{X,S})\to\bar{X}$ are the inclusions):

$$\begin{tikzcd}
 & & 0 \arrow[r, "="] & 0 & \\
 & & \bigoplus\limits_{x\in S} v_{x*}\mathbb{Z} \arrow[u] \arrow[r, "="] & \bigoplus\limits_{x\in S} v_{x*}\mathbb{Z} \arrow[u] & \\
\{1\} \arrow[r] & \mathcal{O}^*_{\bar{X}} \arrow[r] & u_*k(X)^* \arrow[u] \arrow[r, "\mathrm{div}"] & \bigoplus\limits_{x\in\bar{X}} v_{x*}\mathbb{Z} \arrow[u] \arrow[r] & 0 \\
\{1\} \arrow[r] & \mathcal{O}^*_{\bar{X}} \arrow[u, equal] \arrow[r] & w_*\mathcal{O}^*_{\bar{X},S} \arrow[u] \arrow[r, "\mathrm{div}"] & \bigoplus\limits_{x\in X} v_{x*}\mathbb{Z} \arrow[u] \arrow[r] & 0 \\
 & & \{1\} \arrow[u] & \{0\} \arrow[u] &
\end{tikzcd}$$

This right-hand vertical column is obviously exact, while the left-hand column is exact since $\mathcal{O}_{\bar X,S}$ being a semi-local ring has trivial Picard group. Hence the bottom row is an acyclic resolution of $\mathcal{O}^*_{\bar X}$. Now write $I^*_{\bar X,S}$ for the group of rational functions $f\in k(X)^*$ with $f\equiv 1\ (\mathcal{M})$.

We have a diagram:

$$\begin{tikzcd}
 & & \{1\} \arrow[d] & & \\
 & & u_*I^*_{\bar X,S} \arrow[r, "\mathrm{div}"] \arrow[d] & \bigoplus_{x\in X} \upsilon_{x*}\mathbb{Z} \arrow[r] \arrow[d, equal] & \{1\} \\
\{1\} \arrow[r] & \mathcal{O}^*_{\bar X} \arrow[r] \arrow[d] & w_*\mathcal{O}^*_{\bar X,S} \arrow[r, "\mathrm{div}"] \arrow[d] & \bigoplus_{x\in X} \upsilon_{x*}\mathbb{Z} \arrow[r] & \{1\} \\
 & i_*\mathcal{O}^*_S \arrow[r, equal] & i_*\mathcal{O}^*_S \arrow[d] & & \\
 & & \{1\} & &
\end{tikzcd}$$

Since the center column is exact by definition, the hypercohomology of the top row is isomorphic to the hypercohomology of the left-hand column, and so,

$$\mathbb{H}^1(\bar X : \mathcal{O}^*_{\bar X} \to i_*\mathcal{O}^*_S) \simeq \mathbb{H}^1\left(\bar X; u_*I^*_{\bar X,S} \to \bigoplus_{x\in X} \upsilon_{x*}\mathbb{Z}\right) = \mathrm{Pic}(\bar X; S).$$

Corollary 1.2. *There is an exact sequence for n prime to* char(k):

$$H^1_c(X;\mu_n)\to \mathrm{Pic}(\bar X; S) \xrightarrow{n} \mathrm{Pic}(\bar X, S) \to H^2_c(X;\mu_n).$$

Proof. First observe that there is an exact sequence of sheaves on $\bar X$:

$$1\to j_!u_n \to \mu_n \to i_*\mu_n \to 1$$

where $j_!$ is 'extension by zero', so that

$$H^i_c(X;\mu_n) = H^i(\bar X; j_!\mu_n) = \mathbb{H}^i(\bar X; \mu_n \to i_*\mu_n).$$

(cf. [6, Cohomologie étale, IV.5]).

The result now follows from the hypercohomology long exact sequence for the short exact sequence of complexes of sheaves on the étale site:

$$\{1\}\to\left(\begin{array}{c}\mu_n\\ \downarrow\\ i_*\mu_n\end{array}\right)\to\left(\begin{array}{c}\mathbb{G}_m\\ \downarrow\\ i_*\mathbb{G}_m\end{array}\right)\xrightarrow{n}\left(\begin{array}{c}\mathbb{G}_m\\ \downarrow\\ i_*\mathbb{G}_m\end{array}\right)\to\{1\}.$$

Finally, we observe that for $p: X\to T$ a separated morphism of finite type between schemes, if D is a Cartier divisor on X with support proper over T, there is a natural cycle class

$$\gamma(D)\in H^2_{c/T}(X;\mu_n)\overset{\text{def}}{=}H^2(\bar{X};j_!\mu_n)$$

where $j: X\to\bar{X}$ is any compactification of X over T, constructed by first taking the cycle class in $H^2_D(X,\mu_n)$ ([6, Cycle, 2.1.2], using the notation of ibid.) and then taking its image under the natural map (since the support of D is proper over T),

$$H^2_D(X;\mu_n)\to H^2_{c/T}(X;\mu_n).$$

If $D=D_0-D_\infty$ with both D_0 and D_∞ effective, proper and flat over T, then clearly for each point $t\in T$ the cycle class is compatible with restriction to the fibre:

$$\begin{tikzcd}
H^1_D(X;\mathbb{G}_m) \arrow[r, "\gamma"] \arrow[d, "i^*"'] & H^2_{c/T}(X;\mu_n) \arrow[d, "i^*"] \\
H_{D\cap X_t}(X_t;\mathbb{G}_m) \arrow[r, "\gamma"] & H^2_c(X_t;\mu_n)
\end{tikzcd}$$

where $i: X_t\to X$ is the inclusion of the fibre over t.

2. Transfer and generalized Jacobians

As before, let $p: X\to\mathrm{Spec}(k)$ be a nonsingular connected curve over a field k; we do not assume that p is proper.

If $D=\sum_{P\in X}n_P[P]$ is an effective divisor on X, since the local rings of X are discrete valuation rings D uniquely defines a subscheme of X which we also denote D. Let $i_D: D\to X$ be the inclusion, $\pi_D: D\to\mathrm{Spec}(k)$ the natural map (which is finite). For $q\geq 0$, define a map

$$\eta_D: K_q(X;\mathbb{Z}/n)\to K_q(k;\mathbb{Z}/n)$$

by $\eta_D=\pi_{D*}\cdot i_D^*$.

Lemma 2.1. *If $D=D'+D''$ is the sum of two effective divisors, $\eta_D=\eta_{D'}+\eta_{D''}$.*

Proof. This is essentially obvious. Let $\mathbf{M}^1(X)$ be the abelian category of coherent sheaves on X with zero-dimensional support. $p_*:\mathbf{M}^1(X)\to\mathbf{M}(k)$ is an exact functor, and if $D\subset X$ is a divisor, so are $i_{D*}:\mathbf{M}(D)\to\mathbf{M}^1(X)$ and $\pi_{D*}=p_*\cdot i_{D*}:\mathbf{M}(D)\to\mathbf{M}(k)$. Hence η_D is the map induced on K-theory by the composition of exact functors:

$$\mathbf{P}(X)\xrightarrow{i_D^*}\mathbf{M}(D)\xrightarrow{i_{D*}}\mathbf{M}^1(X)\xrightarrow{p_*}\mathbf{M}(k),$$

where $i_{D*}i_D^* = (\ )\otimes_{\mathcal{O}_X}\mathcal{O}_D$. If $D=D'+D''$, then there is an exact sequence:

$$0\to\mathcal{O}_{D'}\to\mathcal{O}_D\to\mathcal{O}_{D''}\to 0$$

and hence by [9, §3, Corollary 3],

$$i_{D*}i_D^* = (i_{D'*}i_{D'}^*) + (i_{D''*}i_{D''}^*) : K_q(X;\mathbb{Z}/n)\to K_q(\mathbf{M}^1(X);\mathbb{Z}/n)$$

and so $\eta_D = \eta_{D'} + \eta_{D''}$.

Lemma 2.2. *Let X be as above, $\bar{X}$, S and $\mathcal{M}$ as in Section* 1. *if D_0, D_∞ are effective (nonzero) divisors on $\bar{X}$ and $D_0 \sim_{\mathcal{M}} D_\infty$,* then $\eta_{D_0} = \eta_{D_\infty}$.

Proof. First suppose D_0 and D_∞ are disjoint.

There is a rational function $f\in k(X)$, $f\equiv 1\ (\mathcal{M})$ with $\mathrm{div}(f) = D_0 - D_\infty$. We may view f as a morphism $f:\bar{X}\to\mathbb{P}_k^1$. Let $\Gamma_f\subset\bar{X}\times\mathbb{P}^1$ be the graph of f. Embed $\mathbb{A}^1$ into $\mathbb{P}^1$ with complement $(1,1)$; then since $f(P)=(1,1)$ for all $P\in S$, we get a diagram, where $\Gamma = \Gamma_f\cap(\bar{X}\times\mathbb{A}^1)$:

$$\begin{tikzcd}
D_0 \arrow[r, "i_0"] \arrow[d, "\theta_0"'] & X\times\{0\} \arrow[r, "p_0"] \arrow[d, "\lambda_0"] & \mathrm{Spec}(k) \arrow[d, "j_0"] \\
\Gamma \arrow[r, "i_\Gamma"] & X\times\mathbb{A}^1 \arrow[r, "\bar{p}"] & \mathbb{A}^1 \\
D_\infty \arrow[r, "i_\infty"] \arrow[u, "\theta_\infty"] & X\times\{\infty\} \arrow[r, "p_\infty"] \arrow[u, "\lambda_\infty"'] & \mathrm{Spec}(k) \arrow[u, "j_\infty"']
\end{tikzcd}$$

Note that Γ is finite and flat over $\mathbb{A}^1$. For $t=0,\infty$, we have Cartesian squares:

$$\begin{tikzcd}
D_t \arrow[r, "\theta_t"] \arrow[d, "\pi_t"'] & \Gamma \arrow[d, "\pi"] \\
\mathrm{Spec}(k) \arrow[r, "j_t"] & \mathbb{A}^1
\end{tikzcd}$$

By [9, §7, Proposition 2.11], for all $q\geq 0$:

$$j_t^*\pi_* = \pi_{t*}\theta_t^* : K_q(\Gamma;\mathbb{Z}/n)\to K_q(k;\mathbb{Z}/n).$$

Hence, by a diagram chase:

$$j_t^* \pi_* i_\Gamma^* = \pi_{t*} i_t^* \lambda_t^* : K_q(X \times \mathbb{A}^1; \mathbb{Z}/n) \to K_q(k; \mathbb{Z}/n).$$

If $q : X \times \mathbb{A}^1 \to X$ is the projection, then $\lambda_t \cdot q = \mathrm{Id}$ for $t = 0, \infty$. As q induces an isomorphism on $K_*(\ ; \mathbb{Z}/n)$, $\lambda_0^* = \lambda_\infty^*$ is the inverse to q^*. Similarly, $j_0^* = j_\infty^*$. Hence if $\alpha \in K_q(X; \mathbb{Z}/n)$:

$$\begin{aligned} \eta_{D_0}(\alpha) &= \pi_{0*} i_0^*(\alpha) = \pi_{0*} i_0^* \lambda_0^* q^*(\alpha) \\ &= j_0^* \pi_* i_\Gamma^* q^*(\alpha) = j_\infty^* \pi_* i_\Gamma^* q^*(\alpha) \\ &= \eta_{D_\infty}(A\alpha) \qquad \text{by symmetry.} \end{aligned}$$

If D_0 and D_∞ are not disjoint, then $D_0 = D_0' + E$, $D_\infty = D_\infty' + E$ where D_0', D_∞' are effective and disjoint, then by from above $\eta_{D_0'} = \eta_{D_\infty'}$ and using Lemma 2.1 we see easily that $\eta_{D_0} = \eta_{D_\infty}$.

Theorem 2.3. *Let* $X, \bar{X}, S, \mathcal{M}$ *be as above. The function*

$$\mathrm{Div}(X) \times K_q(X; \mathbb{Z}/n) \to K_q(k, \mathbb{Z}/n), \qquad (D, \alpha) \mapsto \eta_D(\alpha)$$

induces a pairing

$$\mathrm{Pic}(\bar{X}, S) \otimes_{\mathbb{Z}} K_q(X; \mathbb{Z}/n) \to K_q(k; \mathbb{Z}/n)$$

which factors through a pairing (*for n prime to* Char(k), *and n odd or divisible by four*)

$$H_c^2(X; \mu_n)_{\mathrm{alg}} \otimes_{\mathbb{Z}} K_q(X; \mathbb{Z}/n)) \to K_q(k; \mathbb{Z}/n)$$

where $H_c^2(X; \mu_n)_{\mathrm{alg}} = \mathrm{Pic}(\bar{X}, S) \otimes \mathbb{Z}/n$ *is the image of the divisor class map of Section* 1 *in* $H_c^2(X; \mu_n)$.

Proof. The first assertion follows immediately from Lemmas 2.1 and 2.2, while the second follows from Corollary 1.2.

Remarks. (i) The pairing

$$\mathrm{Pic}(\bar{X}, S) \otimes K_q(X; \mathbb{Z}/n) \to K_q(k; \mathbb{Z}/n)$$

is a special case of a more general pairing, for $X = \bar{X} - S$ separated and of finite type over T, with $\bar{X}$ a compactification of X relative to T and $\Lambda = \mathbb{Z}$ or $\mathbb{Z}/n$:

$$K_p(\bar{X}, S) \otimes K_q(X; \Lambda) \to K_{p+q}(T; \Lambda)$$

where the $K_*(\bar{X}, S)$ are the homotopy groups of the fibre of the natural map of spectra $K(\bar{X}) \to K(S)$. The relationship is given by a spectral sequence (if $\bar{X}$ is regular):

$$E_2^{p,-q} = \mathbb{H}^p(\bar{X}, K_q(\mathcal{O}_{\bar{X}}) \to K_q(\mathcal{O}_S)) \to K_{q-p}(\bar{X}, S)$$

such that if X is a regular connected curve over k:

$$E_2^{1,-1} \simeq \mathrm{Pic}(\bar{X}, S) \simeq K_0(\bar{X}, S).$$

(ii) If $n = 2m$ with m odd, $K_*(\ , \mathbb{Z}/n)$ is $2n$-torsion so we get

$$H_c^2(X, \mu_{2n})_{\mathrm{alg}} \otimes K_q(X, \mathbb{Z}/n) \to K_q(k; \mathbb{Z}/n).$$

3. *K*-theory of separably closed fields

Let k be a separably closed field, F a separably closed extension of k. Fix n prime to char(k).

Theorem 3.1 (Suslin). $K_*(k; \mathbb{Z}/n) \simeq K_*(F; \mathbb{Z}/n)$.

Proof. Since $\mathrm{Spec}(F) = \varprojlim \mathrm{Spec}(E)$, where the limit is over all subfields of F which are finitely generated over k, by [9, 7.2.2], $K_*(F) = \varinjlim K_*(E)$ so we may assume that F is algebraic over a finitely generated extension of k, and hence by induction on the number of generators that F is algebraic over $k(T)$. Further, since $K_*(E, \mathbb{Z}/n) = K_*(F, \mathbb{Z}/n)$ for any purely inseparable extension of F, by [9, §7, 4.7], we may assume that F is the separable closure of $k(T)$, and that k is algebraically closed.

Hence, for all $q \geq 0$:

$$K_q(F; \mathbb{Z}/n) = \underrightarrow{\mathrm{Lim}}\, K_q(U; \mathbb{Z}/n)$$

where the direct limit runs over all connected, étale neighborhoods of the geometric point of $\mathbb{A}_k^1$ determined by $k(T) \subset F$.

Let us write $\varrho_U : \mathrm{Spec}(F) \to U$, $\pi_U : U \to \mathrm{Spec}(k)$, $\pi : \mathrm{Spec}(F) \to \mathrm{Spec(k)}$ for the natural maps. Since every U has a k-rational point $i_x : x = \mathrm{Spec}(k) \to U$ with $\pi_U \cdot i_x = \mathrm{Id}_{\mathrm{Spec}(k)}$, $i_x^* \pi_U^* = \mathrm{Id} : K_q(k; \mathbb{Z}/n) \to K_q(k; \mathbb{Z}/n)$, so $\pi_U^* : K_q(k; \mathbb{Z}/n) \to K_q(U; \mathbb{Z}/n)$ is injective for all U and hence $\pi^* : K_q(k; \mathbb{Z}/n) \to K_q(F; \mathbb{Z}/n)$ is injective. It therefore suffices to show that for all U, $\varrho_U^* = \pi^* i_x^*$ and hence that π^* is surjective. Consider the diagram:

$$\begin{tikzcd}
U \otimes_k F = \tilde{U} \arrow[r, "\tilde{\pi}"] \arrow[d, "f"] & U \arrow[d, "\pi_U"] \\
\mathrm{Spec}(F) \arrow[u, bend left, "g"] \arrow[u, bend right, "j_x"'] \arrow[ur, "\varrho_U"'] \arrow[r, "\pi"'] & \mathrm{Spec}(k) \arrow[u, bend right, "i_x"']
\end{tikzcd}$$

where $\tilde{\pi} \cdot g = \varrho_U$ and $\tilde{\pi} \cdot j_x = i_x \cdot \pi$. Since

$$\varrho_U^* - \pi^* i_x^* = g^* \tilde{\pi}^* - j_x^* \tilde{\pi}^* = (g^* - j_x^*) \cdot \tilde{\pi}^*,$$

it is enough to show that

$$g^* = j_x^* : K_q(\tilde{U}; \mathbb{Z}/n) \to K_q(F; \mathbb{Z}/n).$$

But $g^*-j_x^*=\eta_D$ where $D=[g(\mathrm{Spec}(F))]-[\tilde{\pi}^*x]$ is a divisor of degree zero on the smooth connected curve $\tilde{U}$. By Theorem 2.3, η_D only depends on $\gamma(D)\in H_c^2(\tilde{U};\mu_n)=H^2(\bar{U}\otimes_k F;\mu_n)\simeq\mathbb{Z}/n$ (by [6, VI 2.1]) and since D has degree zero, $\gamma(D)=0$, hence $g^*=j_x^*$.

4. K-theory of Hensel local rings

Fix a positive integer $n\geq 1$.

Theorem 4.1. *Let R be a regular strict Hensel local ring, essentially of finite type over a field k, with $1/n\in R$. If $A=R\{t\}$ is the strict henselization of the local ring at the origin in $\mathbb{A}^1_R$, the map $R\to R\{t\}$ induces an isomorphism for all q:*

$$K_q(R;\mathbb{Z}/n)\simeq K_q(R\{t\};\mathbb{Z}/n).$$

Proof. Let $S=\mathrm{Spec}(R)$, $\bar{x}\in\mathbb{A}^1_S$ the origin in the closed fibre. Then $A=\underrightarrow{\mathrm{Lim}}\,\Gamma(U,\mathcal{O}_U)$ where the direct limit runs over all geometrically pointed connected etale neighborhoods of $\bar{x}$ in $\mathbb{A}^1_S$. Hence

$$K_q(A;\mathbb{Z}/n)=\underrightarrow{\mathrm{Lim}}\,K_q(U;\mathbb{Z}/n).$$

For each étale neighborhood U, the point $\bar{x}$ extends to a section $i_x:\mathrm{Spec}(R)\to U$. If we write $\varrho_U:\mathrm{Spec}(A)\to U$, $\pi_U:U\to\mathrm{Spec}(R)$, $\pi:\mathrm{Spec}(A)\to\mathrm{Spec}(R)$ for the natural maps, arguing as in the proof of Theorem 3.1, we must show that

$$\varrho_U^*=\pi^*i_x^*:K_q(U;\mathbb{Z}/n)\to K_q(A;\mathbb{Z}/n).$$

Consider the diagram, analogous to the diagram of Section 3:

$$\begin{tikzcd}
U\otimes_R A=\tilde{U} \arrow[rr, "\tilde{\pi}"] \arrow[d, "f"'] & & U \arrow[d, "\pi_U"'] \\
\mathrm{Spec}(A) \arrow[u, bend left, "g"] \arrow[u, bend right, "j_x"'] \arrow[urr, "\varrho_U"'] \arrow[rr] & & \mathrm{Spec}(R) \arrow[u, bend right, "i_x"']
\end{tikzcd}$$

As before, it is enough to show that

$$g^*=j_x^*:K_q(\tilde{U};\mathbb{Z}/n)\to K_q(A;\mathbb{Z}/n).$$

Let F be the field of fractions of A; note that F is not separably closed, and write $\phi:\mathrm{Spec}(F)\to\mathrm{Spec}(A)$ for the natural map. Since R and hence A is assumed to be essentially of finite type over a field, A is a direct limit of local rings $\mathcal{O}_{Y,y}$ of regular points on varieties of finite type over k. If $k(Y)$ is the field of fractions of $\mathcal{O}_{Y,y}$, then, by [9, §7, 5.1], $K_q(\mathcal{O}_{Y,y};\mathbb{Z}/n)$ injects into $K_q(k(Y);\mathbb{Z}/n)$. Hence, passing to the direct limit, $\phi^*:K_q(A;\mathbb{Z}/n)\to K_q(F;\mathbb{Z}/n)$ is injective.

Now consider the diagram, where f is a morphism of finite type with fibres curves:

$$\begin{tikzcd}
\tilde{U}_F \arrow[r, "\tilde{\phi}"] \arrow[d] & \tilde{U} \arrow[d, "f"] & \tilde{U}_0 \arrow[l, "\tilde{\psi}"'] \arrow[d, "f_0"] \\
\mathrm{Spec}(F) \arrow[u, bend left, "g_F"] \arrow[u, bend right, "j_{x,F}"'] \arrow[r, "\phi"'] & \mathrm{Spec}(A) \arrow[u, bend left, "g"] \arrow[u, bend right, "j_x"'] & \mathrm{Spec}(L) \arrow[l, "\psi"] \arrow[u, bend left, "g_0"] \arrow[u, bend right, "j_{x,0}"']
\end{tikzcd}$$

where L is the residue field of A. Since

$$(g_F^* - j_{x,F}^*)\cdot\tilde{\phi}^* = \phi^*(g^* - j_x^*): K_q(\tilde{U}; \mathbb{Z}/n) \to K_q(F; \mathbb{Z}/n)$$

and ϕ^* is injective, it is enough to show that $g_F^* - j_{x,F}^* = 0$. Now observe that since g and j_x are sections of f, $D = [g(\mathrm{Spec}(A))] - [j_x(\mathrm{Spec}(A))]$ is a divisor on $\tilde{U}$ with support proper (in fact finite) over $\mathrm{Spec}(A)$, and that if D_F is the restriction of D to the generic fibre $\tilde{U}_F$, then

$$g_F^* - j_{x,F}^* = \eta_{D_F}.$$

If $\gamma(D) \in H^2_{c/S}(\tilde{U}; \mu_n)$ is the cycle class constructed in Section 1, then since A is the strict Henselization of $\mathcal{O}_{\mathbb{A}^1_R, x}$, g and j_x coincide when restricted to the closed fibre $\tilde{U}_0$, hence

$$\tilde{\psi}^*(\gamma(D)) = 0 \in H^2_c(\tilde{U}_0; \mu_n).$$

But by proper basechange [6, (IV, 5.4)], $\tilde{\psi}^*$ is an isomorphism, hence $\gamma(D) = 0$. Now by Theorem 2.3, η_{D_F} only depends on $\gamma(D_F) = \tilde{\phi}^*\gamma(D) \in H^2_c(\tilde{U}_F; \mu_n)$ and hence is zero.

Corollary 4.2. *Let R be the strict Henselization of the local ring at a smooth point of a variety of finite type over a separably closed field k. Then for $(n, \mathrm{char}(k)) = 1$, the map $k \to R$ induces an isomorphism:*

$$K_*(R; \mathbb{Z}/n) \simeq K_*(k; \mathbb{Z}/n).$$

Proof. By the Cohen–Seidenberg therorem, EGA O_{IV}, 19.6.6, there is a finite radicial extension k'/k such that the completion of $R' = R \otimes_k k'$ is isomorphic over k' to a formal power series ring $F'[[t_1, t_2, \ldots, t_m]]$. Here $F' = F \otimes_k k'$ for F the separably closed residue field of R. By Artin approximation, [1, 2.6], the strict hensel local ring R' is isomorphic over k' to $F'\{t_1, \ldots, t_m\}$. By Theorem 4.1, induction on m, and Theorem 3.1, the obvious maps induce isomorphisms

$$K_*(R'; \mathbb{Z}/n) \cong K_*(F'\{t_1, \ldots, t_m\}; \mathbb{Z}/n) \cong K_*(F'; \mathbb{Z}/n) \cong K_*(k'; \mathbb{Z}/n).$$

The finite radicial extensions R'/R, k'/k induce isomorphisms on $K_*(\ ; \mathbb{Z}/n)$ for n prime to the characteristic by [9, §7, 4.7]. Combining these isomorphisms with those above shows that $k \to R$ induces an isomorphism on $K_*(\ ; \mathbb{Z}/n)$.

Note 4.3. Gabber points out that the proof of 4.1 does not use the strictness of R. Similarly, the proof of 4.2 shows that if R is the (unstrict) Henselization of the local ring at a smooth point of a variety, and F is the residue field of R, then the quotient map $R \to F$ induces an isomorphism $K_*(R; \mathbb{Z}/n)) \to K_*(F; \mathbb{Z}/n)$. Gabber has generalized this to the case of non equi-characteristic Hensel pairs.

Corollary 4.4. *Let X be a scheme smooth over a separably closed field k, or over a geometrically regular strict local Hensel ring essentially of finite type over such a k. Then the sheaf $K_*(\ ; \mathbb{Z}/n)$ on the étale site of X is the constant sheaf isomorphic to $K_*(k; \mathbb{Z}/n)$.*

Proof. It suffices to show that the natural map $K_*(k; \mathbb{Z}/n) \to K_*(\ ; \mathbb{Z}/n)$ induces an isomorphism at every stalk; i.e., when evaluated at every strict Hensel local ring of X. But this holds by Corollary 4.2.

5. Stable homotopy of $\mathbb{C}P^\infty$ and $K_*(\mathbb{C})$

When combined with Quillen's calculation of the K-groups of the algebraic closure of a finite field, Suslin's theorem of Section 3 yields a calculation of the groups $K_*(k; \mathbb{Z}/n)$ for all separably closed fields k of positive characteristic prime to n; these are isomorphic to $\pi_*(BU; \mathbb{Z}/n)$ for $* > 0$. We have recently heard that Suslin can prove this in characteristic 0. Here we obtain independently partial results in this direction. By Theorem 3.1 we need only consider the case of the complex numbers $\mathbb{C}$.

Theorem 5.1. *The* mod n *stable homotopy groups of $\mathbb{C}P^\infty$ surject onto $K_*(\mathbb{C}; \mathbb{Z}/n)$ for $* \geq 0$.*

Proof. The idea is to use the splitting principle technique of [12].

We will concentrate on the parts of the argument that differ from that of [12], the read may consult this reference for aid in completing the sketch.

For A a ring, the inclusion of the A-points of the normalizer of a maximal torus in GL_N induces a map:

$$B\left(\Sigma_N \int \mathrm{GL}_1(A)\right)^+ \to B\mathrm{GL}_N(A)^+. \tag{5.1}$$

This extends to a map of infinite loop spaces, or of spectra

$$\gamma : \Sigma^\infty(B\mathrm{GL}_1(A)) \to K(A). \tag{5.2}$$

For A a strict Hensel local ring over $\mathbb{Z}[1/n]$, $B\mathrm{GL}_1(A)$ and $\mathbb{C}P^\infty$ have homotopy equivalent n-adic completions, so that there are isomorphisms

$$\begin{aligned}\pi_*(\Sigma^\infty/n(B\mathrm{GL}_1(A))) &\simeq \pi_*(\Sigma^\infty/n(\mathbb{C}P^\infty))\\ &\simeq \pi_*(\Sigma^\infty(\mathbb{C}P^\infty);\mathbb{Z}/n)\\ &\simeq \pi_*^s(\mathbb{C}P^\infty;\mathbb{Z}/n).\end{aligned}\tag{5.3}$$

In particular, for $A=\mathbb{C}$, the mod n reduction of (5.2) is the map that the theorem claims is a surjection.

For X a scheme smooth over $\mathbb{C}$, one has the isomorphism of Corollary 4.4. Together with γ and the isomorphisms of (5.3), this gives a diagram of étale hypercohomology spectra

$$\begin{tikzcd}
\mathbb{H}^{\cdot}_{\mathrm{et}}(X;\Sigma^\infty/n(B\mathrm{GL}_1(\;))) \arrow[r, "\gamma"] & \mathbb{H}^{\cdot}_{\mathrm{et}}(X;K/n(\;)) \\
\mathbb{H}^{\cdot}_{\mathrm{et}}(X;\Sigma^\infty/n(\mathbb{C}P^\infty)) \arrow[r, "\gamma"] \arrow[u, "\simeq"] & \mathbb{H}^{\cdot}_{\mathrm{et}}(X;K/n(\mathbb{C})). \arrow[u, "\simeq"']
\end{tikzcd}\tag{5.4}$$

Using Weibel's excision theorem as in [12], one sees that diagram (5.4) also exists if X is smooth over a union of complex hyperplanes in the configuration of a simplicial complex.

The diagram (5.4) allows us to switch back and forth between the half-algebraic γ on the top and the purely topological γ on the bottom. We use this frequently without further comment.

As the étale and classical topologies agree cohomologically for any complex variety X, the bottom map in (5.4) is the map of classical generalized cohomology theories induced by the map of coefficient spectra $\gamma:\Sigma^\infty/n(\mathbb{C}P^\infty)\to K/n(\mathbb{C})$.

Let $\mathcal{E}$ be a vector bundle over X. Then $\mathcal{E}$ corresponds to a GL_N-torsor over X. Let $p:X'\to X$ be the quotient of this torsor by the action of the normalizer of the maximal torus $N(T)=\Sigma_N\int\mathrm{GL}_1$. Then $p^*\mathcal{E}$ admits reduction of structure group from GL_N to $N(T)$. As the Euler characteristic of the fibre $\mathrm{GL}_N/N(T)$ is $\{1\}$ (cf. [3, §6]), the Becker–Gottlieb transfer of [4] naturally splits the mononomorphism induced by $p:X'\to X$ on any generalized cohomology theory.

If X is a union of hyperplanes in the configuration of a simplicial complex, e.g., if X is a mod n Moore space over $\mathbb{C}$, comparison of the Mayer–Vietoris spectral sequence of Dayton and Weibel [5] for $K(\;;\mathbb{Z}/n)$ and the hypercohomology spectral sequence for $\mathbb{H}^{\cdot}_{\mathrm{et}}(X;K/n(\mathbb{C}))$ shows that there are isomorphisms

$$K_*(X;\mathbb{Z}/n)\simeq\pi_*\mathbb{H}^{\cdot}_{\mathrm{et}}(X;K/n(\mathbb{C})).\tag{5.5}$$

Let $x\in K_i(\mathbb{C};\mathbb{Z}/n)$. We wish to show that it is in the image of γ. The cases $i\leq 1$ are easy, so assume $i\geq 2$. Then x corresponds to an element of augmentation 0 in $K_0(Y;\mathbb{Z}/n)$ for Y a simplicial i-sphere over $\mathbb{C}$, and to an x of augmentation 0 in $K_0(X)$, for X a mod n Moore space over $\mathbb{C}$. As in [12], it suffices to show that the mod n reduction of x in $K_0(X;\mathbb{Z}/n)$ is in the image of the bottom γ of (5.4). Let $x\in K_0(X)$ be represented by a rank-N vector bundle $\mathcal{E}$ minus a rank-N trivial bundle. Let $p: X' \to X$ be the $\mathrm{GL}_N/N(T)$-bundle associated to $\mathscr{E}$. By the Becker-Gottlieb natural splitting, it suffices to prove $p^*(x)$ is in the image of γ on X . But on X', $p^*\mathscr{E}$ admits reduction of structure group along $\gamma: \Sigma_n \int \mathrm{GL}_1 \to \mathrm{GL}_n$, so $p^*(x)$ is in the image of the top map γ of (5.4). This proves the theorem.

Corollary 5.2. *Let l^ν be a prime power. The for $0 \leq i \leq 2l-4$ one has*

$$K_i(\mathbb{C}; \mathbb{Z}/l^\nu) = \begin{cases} \mathbb{Z}/l^\nu, & i \text{ even}, \\ 0, & i \text{ odd} \end{cases} \tag{5.6}$$

For all i, $K_i(\mathbb{C}; \mathbb{Z}/l^\nu)$ is finite.

Proof. The Dwyer-Friedlander map detects quotients of $K_*(\mathbb{C}; \mathbb{Z}/l^\nu)$ that are this big. Theorem 5.1 and calculations of the stable homotopy of $\mathbb{C}P^\infty$ by Liulevicius [7] or the delicate Hurewicz theorem give these values as an upper bound in this range.

The last statement is immediate from Theorem 5.1 as the stable homotopy groups of $\mathbb{C}P^\infty$ are finitely generated.

Corollary 5.3. *Let $K_*(\mathbb{C}; \mathbb{Z}_l^\wedge)$ denote the homotopy groups of the l-adic completion of the spectrum $K(\mathbb{C})$. Then there is an isomorphism*

$$K_i(\mathbb{C}; \mathbb{Z}_l^\wedge) = \varprojlim_\nu K_i(\mathbb{C}; \mathbb{Z}/l^\nu). \tag{5.7}$$

There is a surjection

$$\pi_i^S(\mathbb{C}P^\infty) \otimes \mathbb{Z}_l^\wedge \to K_i(\mathbb{C}; \mathbb{Z}_l^\wedge). \tag{5.8}$$

Proof. As the groups $K_*(\mathbb{C}, \mathbb{Z}/l^\nu)$ are finite, the inverse system as ν increases satisfies the Mittag-Leffler condition. Thus $\varprojlim^1 = 0$, and (5.7) drops out.

The groups $\pi_*^S(\mathbb{C}P^\infty)$ are finitely generated, so the system of kernels of the maps $\pi_*^S(\mathbb{C}P^\infty; \mathbb{Z}/l^\nu) \to K_j(\mathbb{C}; \mathbb{Z}/l^\nu)$ also satisfies Mittag-Leffler. The surjectivity of (5.8) now follows as the inverse limit of Theorem 5.1.

Corollary 5.4. *For all $i \geq 0$,*

$$K_i(\mathbb{C} : \mathbb{Z}_l^\wedge) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}_l^\wedge, & i \textit{ even}, \\ 0, & i \textit{ odd}. \end{cases} \tag{5.9}$$

Proof. The l-adic Dwyer-Friedlander map detects these quotients. The $\mathbb{Q}_l^\wedge$-stable homotopy of $\mathbb{C}P^\infty$ gives these as an upper bound.

Corollary 5.5. *The discrete group* $\mathrm{GL}(\mathbb{C})$ *has as $\mathbb{Q}_l^\wedge$-cohomology a polynomial ring in the Chern classes*

$$(\varprojlim_\nu H^*(\mathrm{GL}(\mathbb{C}); \mathbb{Z}/l^\nu)) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \mathbb{Q}_l^\wedge[c_1, c_2, \ldots].$$

Proof. Immediate from Corollary 5.4.

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