Such cases as figures 2, 3, etc., need a much larger standard of comparison, L; but these and other results are improved by the use of a small inductor (say 0.3 hen. in the secondary) and radio telephones. Omitting these, there is room here for Graph 4, obtained with the mere exchange of resistances, which merits some further attention. If R is the fixed resistance commutated, R' and R'' the counter values corresponding to positions I and II, since internally $L_0 = L_0'$, and $R_0 = R_0'$ the equations reduce to

$$c + c' = c \frac{\sqrt{(R + R_0)^2 + L_0^2\omega^2}}{\sqrt{(R'' + R_0)^2 + L_0^2\omega^2}} + c' \frac{\sqrt{(R + R_0)^2 + L_0^2\omega^2}}{\sqrt{(R' + R_0)^2 + L_0^2\omega^2}}.$$

A solution of this equation is $R' = R'' = R$, so that the paired curves of figure 4 intersect near $R = 0$, $R = 100$, $R = 500$ ohms (the case $R = 100$ left without reversal).

The case of $\Delta s' = 0$ and $\Delta s'' = 0$ for the two positions I and II, is available for $R = 100$ ohms. The other cases ($R = 0$ and $R = 500$) do not reach the abscissa. We thus have again

$$\frac{c}{c'} = \frac{\sqrt{(R + R_0)^2 + L_0^2\omega^2}}{\sqrt{(R' + R_0)^2 + L_0^2\omega^2}} = \frac{\sqrt{(R'' + R_0)^2 + L^2\omega^2}}{\sqrt{(R + R_0)^2 + L^2\omega^2}}.$$

If we insert the values $R = 100$, $R' = 50$, $R'' = 150$ ohms, as given by the graphs and the constants $L_0 = 0.06$ and $R_0 = 84$, we obtain $c/c' = 1.16$, in both cases, which agrees very well with $c/c' = 1.19$ deduced from inductances, in the preceding section.

A STATISTICAL QUANTUM THEORY OF REGULAR REFLECTION AND REFRACTION[1]

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The success of the quantum theory and more particularly of that view of the quantum theory which regards the energy of the quantum as highly localized has made it seem important to various authors to attempt an explanation on the basis of that view of some of the phenomena of optics which have been regarded as typical wave phenomena. In the following discussion, only a statistical treatment will be attempted; i.e., only large aggregates of quanta will be considered and the media traversed by these quanta will be regarded as continuous. The first of these restrictions is rendered advisable by the nature of the phenomena; the terms appropriate to the discussion of reflection, refraction and radiation pressure are scarcely intelligible if applied to only a few quanta. The second restriction is supported as regards quanta of low energy by the view widely held that such quanta occupy volumes large in comparison with intra-atomic distances, so that many electrons may be included in the volume occupied by one quantum. Compact quanta of large energy, such as those considered as acting in the Compton effect, are not contemplated in this discussion.

It is assumed at the outset that the quantum traverses a medium with the velocity of light in that medium. It is further assumed that the change in velocity of a quantum as it passes from one medium to another is not accompanied by a change in its energy.

The view is here adopted that the boundary surface between two transparent media has work done upon it by the incident radiation, and that it does work upon reflected or refracted radiation. By the principle of the conservation of energy the work in a given time done upon the surface is then equal to the work done by the surface. Consider a beam of unit cross-section incident normally upon such a surface. Let n, n' and n'' be the numbers of quanta per second in the incident, reflected and refracted beams respectively. We then have by the conservation of energy

$$nh\nu = n'h\nu + n''h\nu \qquad (1)$$

It is now assumed that the relation of the momentum to the energy in each beam is such that the work done upon the surface by the incident beam is proportional to pv_1, that done by the surface upon the reflected beam to $p'v_1$, and that upon the refracted beam to $p''v_2$, where p, p' and p'' are the respective pressures or momenta per square centimeter per second exerted by the incident, reflected and refracted radiation upon the surface, and v_1 and v_2 are the respective velocities of light in the media 1 and 2.

We then have from (1)

$$nh\nu/pv_1 = n'h\nu/p'v_1 = n''h\nu/p''v_2 = k \qquad (2)$$

where k is a constant. If the medium 1 is a vacuum, $nh\nu/pc = k = 1$, since $nh\nu/c = p$. Accordingly, $n'h\nu/v_1 = p'$ and $n''h\nu/v_2 = p''$, or, in general,

$$nh\nu/v = p. \qquad (3)$$

Thus the momentum of a linear stream of radiation in any medium is equal to the energy density of the radiation. From this result it is possible with the principle of least action to derive Fermat's principle; with the principle of the conservation of momentum to derive the laws of radiation pressure; and with the principle of the conservation of energy to derive Fresnel's equation for the relative intensities of the incident, reflected and refracted beams.

Let us write the principle of least action in the form

$$\delta \int_a^b M\,ds = 0$$

where M is the momentum and ds is the element of distance along the path from a to b. Replacing M by $h\nu/v$, we have for the quantum:

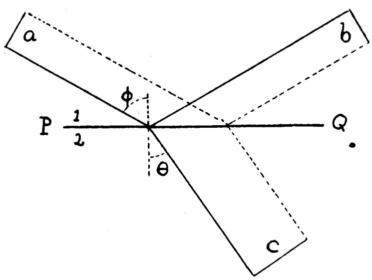
$$\delta \int_a^b \frac{h\nu}{v}\,ds = 0.$$

Replacing ds/v by dt and dividing by $h\nu$, the energy of the quantum, assumed constant, we have:

$$\delta \int_a^b dt = 0, \quad \text{or,} \quad \delta(t_b - t_a) = 0 \tag{4}$$

which is the mathematical expression of Fermat's principle. Since the laws of the angles of incidence, reflection, and refraction are consequences of Fermat's principle, we are now enabled to use these laws to derive the laws of radiation pressure.

In the figure, let PQ represent the boundary between two media, 1 and 2. Let the velocity of light in the first medium be v_1, in the second medium be v_2. Let a, b and c be, respectively, the incident, reflected, and refracted beams, the angle of incidence being ϕ and the angle of refraction being θ. Let the area of the portion of the boundary intercepted by the beams be unity. Let n be the number of quanta crossing unit area of the cross-section of the incident beam in unit time. Then $nh\nu$ will be the intensity of the incident beam. Similarly let $n'h\nu$ be the intensity of the reflected beam and $n''h\nu$ be the intensity of the refracted beam. The number of quanta incident in unit time on unit area of the boundary will be $n \cos \phi$, the number reflected from unit area will be $n' \cos \phi$, the number refracted through unit area will be $n'' \cos \theta$. The normal component of the momentum of the quanta incident in unit time on unit area of the surface will be $n \frac{h\nu}{v_1} \cos^2 \phi$, of the quanta reflected will be $-n' \frac{h\nu}{v_1} \cos^2 \phi$, of the quanta refracted will be $n'' \frac{h\nu}{v_2} \cos^2 \theta$, the positive direction being taken as downward. By the principle of the conservation of momentum the radiation pressure on the boundary will be



$$p = n\frac{h\nu}{v_1}\cos^2\phi + n'\frac{h\nu}{v_1}\cos^2\phi - n''\frac{h\nu}{v_2}\cos^2\theta. \qquad (5)$$

Replacing $nh\nu$ by E, the intensity of the incident beam, $n'h\nu$ by R, the intensity of the reflected beam, and $n''h\nu$ by D, the intensity of the refracted beam, we have:

$$p = \frac{E+R}{v_1}\cos^2\phi - \frac{D}{v_2}\cos^2\theta. \qquad (6)$$

By exactly similar reasoning, we have for the tangential force per unit area on the boundary (the positive direction being taken to the right):

$$T = \frac{E-R}{v_1}\cos\phi\sin\phi - \frac{D}{v_2}\cos\theta\sin\theta. \qquad (7)$$

These are identical with the expressions derived from the wave theory and verified experimentally for various cases by Lebedew, Nichols and Hull, Poynting, and others. (Cf. J. H. Poynting, *Phil. Mag.*, **9**, pp. 169–171 and 393–406, 1905.)

By the principle of the conservation of energy and the constancy of $h\nu$, the number of quanta reaching the surface in unit time is equal to the number leaving the surface in unit time. That is:

$$n\cos\phi = n'\cos\phi + n''\cos\theta \qquad (8)$$

whence, since n, n' and n'' are respectively proportional to E, R and D,

$$\frac{E-R}{D} = \frac{\cos\theta}{\cos\phi}. \qquad (9)$$

This is equivalent to Fresnel's equation:

$$\frac{a^2-b^2}{c^2} = \frac{\tan\phi}{\tan\theta}$$

where a, b and c are, respectively, the amplitudes of the incident, reflected and refracted beams; for the intensity of the beam is directly proportional to the square of the amplitude and inversely proportional to the velocity of propagation. Hence:

$$\frac{v_1(E-R)}{v_2D} = \frac{\tan\phi}{\tan\theta}$$

or:

$$\frac{E-R}{D} = \frac{v_2\tan\phi}{v_1\tan\theta} = \frac{\sin\theta\tan\phi}{\sin\phi\tan\theta} = \frac{\cos\theta}{\cos\phi}$$

as above.

The essential point of the foregoing discussion is that the laws of reflection and refraction of light are inherent in the ascription to the quantum of the momentum $h\nu/v$ without the use of any other postulate. As to the extent that the mechanism of radiation is determined by this assumption the authors have reached no definite conclusion. They believe, however, that a study of the consequences of this assumption may yield information on the nature of the quantum itself.

[1] The substance of this paper was presented before the American Physical Society, April 25, 1925. Abstract, *Phys. Rev.*, **25**, p. 896, June, 1925.

THE DISTRIBUTION OF DIFFRACTING POWER IN SODIUM CHLORIDE

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Duane (cf. preceding note, page 489) has applied Epstein and Ehrenfest's[2] quantum treatment of the problem of Fraunhofer diffraction to the determination of the distribution of diffracting power in a crystal. The use of his ideas leads to the following expression for the "electron density" at a point (xyz) in the unit cell of a crystal of NaCl:

$$\rho_{(xyz)} = \sum_{n_1} \sum_{n_2} \sum_{n_3} A_{n_1n_2n_3} \cos 2\pi n_1 x/a \cos 2\pi n_2 y/a \cos 2\pi n_3 z/a \qquad (1)$$

where n_1, n_2, n_3 are the Miller indices of the different crystal planes multiplied by the number representing the order of reflection; a is the length of side of the unit cell; $A_{n_1n_2n_3}$ is the square root of the intensity of the reflection from the plane ($n_1n_2n_3$), and has the same value for all combinations (plus and minus) of these numbers. The expression "electron density" will be used for the purpose of brevity, but it must be remembered that what is obtained from (1) is not necessarily the actual distribution of electrons, but rather the distribution of diffracting power.

In order to determine the electron density at a point in a crystal of NaCl, it is necessary to have experimental values for the A's out to fairly high values of $n_1n_2n_3$. The best measurements of these quantities have been made by Bragg, James and Bosanquet.[3] Their theoretical expression for the intensity of reflection of X-rays at an angle θ from a single crystal face is:

$$I \propto \frac{N^2 f^2 \lambda^3 (1 + \cos^2 2\theta)\, e^{-\frac{b \sin^2 \theta}{\lambda^2}}}{F(\mu) \sin 2\theta} \qquad (2)$$

N is the number of electrons per unit volume; f is a function of $\dfrac{\lambda}{\sin \theta}$ and