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CONTROL THEORY

A Canonical Form of Vector Control Systems

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1. INTRODUCTION

We consider the linear stationary system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the phase vector of the system; $y, u \in \mathbb{R}^l$ are the output and input vectors, respectively (we assume that the dimensions of y and u coincide, i.e., the system is square); and A, B, and C are constant matrices of the corresponding size. We assume that the pair $\{A; B\}$ is controlled and the pair $\{C; A\}$ is observed, i.e., the system is in general position.

There exist canonical representations of system (1) convenient for solving particular problems. Thus, in solving the stabilization problem for a scalar system of the form (1), it is convenient to reduce the system to a canonical form with zero dynamics separated out. The purpose of this paper is to obtain a similar canonical form for vector systems and describe an algorithm for reducing systems to this form. Such a representation is important because it substantially simplifies solving the stabilization and identification problems (in particular, for linear systems under the uncertainty conditions).

2. SCALAR SYSTEMS

First, consider the well-studied case of a scalar system, in which $l = 1$. A generic scalar system of the form (1) can be reduced by a nondegenerate transformation of coordinates [1] to the canonical controllability form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= -a_1x_1 - \ldots - a_nx_n + u; \\ y &= c_1x_1 + \ldots + c_nx_n. \end{aligned} \tag{2}$$

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A motion in system (1) which is contained entirely in the manifold $y = Cx = 0$ is traditionally referred to as a zero dynamics of system (1). For a linear stationary system, a zero dynamics is described by a system of linear stationary equations. For such a system, a characteristic polynomial is defined; in what follows, we call it the characteristic polynomial of zero dynamics.

For square systems, the characteristic polynomial of zero dynamics coincides with the determinant of the Rosenbrock matrix [2, 3]:

$$\beta(s) = \det R(s) = \det\left[\begin{array}{c|c} sI - A & -B \\ \hline C & 0 \end{array}\right]. \tag{3}$$

The characteristic polynomial of zero dynamics is especially easy to find for scalar systems. In this case, the transfer function

$$W(s) = C(sI - A)^{-1}B = \frac{\beta(s)}{\alpha(s)} \tag{4}$$

is defined, where $\alpha(s)$ and $\beta(s)$ are polynomials in s, $\deg\alpha(s) = n$, and $\deg\beta(s) < n$. The polynomial $\alpha(s)$ is the characteristic polynomial of the matrix A, and $\beta(s)$ is the characteristic polynomial of zero dynamics. For generic scalar systems, the relative order of the system is defined; this is the number r such that

$$CB = 0, \quad CAB = 0, \ldots, CA^{r-2}B = 0,$$
$$CA^{r-1}B \neq 0,$$

and $\deg\beta(s) = n - r$. It follows from the definition of relative order that the first $r - 1$ time derivatives of the output $y(t)$ do not explicitly depend on the input $u(t)$, while $y^{(r)}(t)$ depends on $u(t)$ explicitly; to be more precise,

$$y^{(r)} = CA^rx + CA^{r-1}Bu.$$

Note also that, in the canonical form (2), the a_i and c_j are the coefficients of the polynomials $\alpha(s)$ and $\beta(s)$, respectively; i.e., we have

$$\alpha(s) = s^n + a_ns^{n-1} + \ldots + a_1,$$

$$\beta(s) = c_ns^{n-1} + c_{n-1}s^{n-2} + \ldots + c_1,$$

where the leading coefficients c_j may be zero.

Suppose that the relative order of the system equals r. Without loss of generality, we can assume that $CA^{r-1}B = 1$ (this can always be achieved by normalizing the output $y(t)$). Then,

$$\beta(s) = s^{n-r} + c_{n-r}s^{n-r-1} + \ldots + c_1,$$

where $c_{n-r+1} = CA^{r-1}B = 1$. In this case, to reduce system (2) to a canonical representation with zero dynamics separated out, we pass to the coordinates

$$x' = \begin{cases} x_1 \\ \vdots \\ x_{n-r}; \end{cases}$$

$$y' = \begin{cases} y_1 = y = Cx \\ y_2 = \dot{y} = CAx \\ \ldots \\ y_r = y^{(r-1)} = CA^{r-1}x. \end{cases}$$

The transition matrix from the coordinates x to the coordinates $\begin{pmatrix} x' \\ y' \end{pmatrix}$ has the form

$$P = \left(\begin{array}{cccc|cccc} 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & & & \vdots & & & \vdots \\ 0 & 0 & & 1 & 0 & 0 & \ldots & 0 \\ \hline c_1 & c_2 & \ldots & c_{n-r} & 1 & 0 & \ldots & 0 \\ 0 & c_1 & \ldots & c_{n-r-1} & c_{n-r} & 1 & \ldots & 0 \\ & & & & & & \ddots & \\ 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \end{array}\right),$$

$$\det P = 1.$$

In the new coordinates, the equation of the system takes the form

$$\begin{cases} \dot{x}_1 = x_2 \\ \quad\vdots \\ \dot{x}_{n-r-1} = x_{n-r} \\ \dot{x}_{n-r} = -c_1x_1 - c_2x_2 - \ldots - c_{n-r}x_{n-r} + y, \end{cases}$$

$$\begin{cases} \dot{y}_1 = y_2 \\ \quad\vdots \\ \dot{y}_{r-1} = y_r \\ \dot{y}_r = -\sum_{i=1}^{n-r}\alpha_i x_i - \sum_{j=1}^{r}\gamma_j y_j + u, \end{cases} \tag{5}$$

$$y = y_1.$$

System (5) can be written in the compact form

$$\begin{aligned} \dot{x}' &= A_{11}x' + A_{12}y, \\ \dot{y}_1 &= y_2, \\ &\vdots \\ \dot{y}_{r-1} &= y_r, \\ \dot{y}_r &= -\bar{\alpha}x' - \bar{\gamma}y' + u, \end{aligned} \tag{5'}$$

where A_{11} is a companion matrix of the polynomial $\beta(s)$, $\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-r})$, $\bar{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_r)$, and $A_{12} = (0, 0, \ldots, 0, 1)^T$.

The representation of the system in the form (5') is convenient because the equation of zero dynamics can be written in the form

$$\dot{x}' = A_{11}x';$$

it is determined by the initial $(n-r)$-dimensional part of the system, while the remaining coordinates are derivatives of the output.

For the coefficients α_i and γ_j, the following assertion is valid.

Statement 1. *Suppose that a generic system of the form* (1) *with scalar input and output has transfer function* $W(s) = C(sI - A)^{-1}B = \dfrac{\beta(s)}{\alpha(s)}$, *where* $\alpha(s)$ *and* $\beta(s)$ *are coprime polynomials*, $\deg\alpha(s) = n$, $\deg\beta(s) = m$, *and the leading coefficients of these polynomials are equal to* 1. *Suppose also that polynomials* $\varphi(s)$ *and* $\psi(s)$ *are defined as the quotient and remainder on division of* $\alpha(s)$ *by* $\beta(s)$:

$$\frac{\alpha(s)}{\beta(s)} = \varphi(s) + \frac{\psi(s)}{\beta(s)}, \quad \deg\varphi(s) = n - m = r,$$

$$\deg\psi(s) < m = n - r.$$

Then, the α_i *and* γ_i *in the canonical representation* (5) *are the coefficients of the polynomials* $\psi(s)$ *and* $\varphi(s)$, *respectively, i.e.*,

$$\begin{aligned} \varphi(s) &= s^r + \gamma_r s^{r-1} + \ldots + \gamma_1, \\ \psi(s) &= s^{n-r-1}\alpha_{n-r} + \ldots + \alpha_1. \end{aligned} \tag{6}$$

Proof. Suppose that the polynomials $\varphi(s)$ and $\psi(s)$ have the form (6), where the α_i and γ_i are the coefficients in the canonical form (5). Let us show that they are, respectively, the quotient and remainder on division of $\alpha(s)$ by $\beta(s)$. For this purpose, we perform the Laplace transform for system (5') under zero initial conditions. Using the same notation for the Laplace images of coordinates as for the preimages, we obtain

$$(sI - A_{11})x' = A_{12}y \Rightarrow x' = (sI - A_{11})^{-1}A_{12}y.$$

The last relation in (5') implies

$$s^r y = -\bar{\alpha}x' - \sum_{j=1}^{r} \gamma_j s^{j-1} y + u.$$

Therefore,

$$(s^r + \gamma_r s^{r-1} + \ldots + \gamma_1)y + \bar{\alpha}x'$$
$$= \varphi(s)y + \bar{\alpha}(sI - A_{11})^{-1}A_{12}y = u.$$

Taking into account the explicit representation for A_{11} and A_{12}, we obtain

$$\bar{\alpha}(sI - A_{11})^{-1}A_{12} = \frac{\psi(s)}{\beta(s)},$$

whence

$$\varphi(s)y + \frac{\psi(s)}{\beta(s)}y = u.$$

Since $y = \dfrac{\beta(s)}{\alpha(s)}u$, it follows that $u = \dfrac{\beta(s)}{\alpha(s)}y$. Therefore, $\varphi(s)$ is the quotient on division of $\alpha(s)$ by $\beta(s)$, and $\psi(s)$ is the remainder.

This completes the proof of the statement.

3. VECTOR SYSTEMS

In this section, we describe an algorithm for reducing a vector system to a canonical form similar to (5'). We assume that system (1) is generic and the dimensions of the input and output coincide (i.e., the system is square). Moreover, we assume that, for system (1), a vector order in the sense of Isidori is defined.

Definition (Isidori). A vector $r = (r_1, r_2, \ldots, r_l)$ is called a vector relative order of system (1) if the following conditions hold:

(i) $c_i A^{r_i - 1} B \neq 0$; for all $i = 1, 2, \ldots, l$ and $j = 0, 1, \ldots, r_{i-2}$, $c_i A^j B = 0$;

(ii) $\det H(r_1, r_2, \ldots, r_l) = \det \begin{bmatrix} c_1 A^{r_1 - 1} B \\ \ldots \\ c_l A^{r_l - 1} B \end{bmatrix} \neq 0,$

where the c_i are the rows of the matrix C.

Condition (i) in this definition means that the derivatives of the output $y_i = c_i x$ up to order $r_i - 1$ do not explicitly depend on the input u, while the r_ith derivative depends on u explicitly.

Condition (ii) means that, for the control $u(t) \in \mathbb{R}^l$, the matrix $H(r)$ in the equations for the r_ith derivatives of the input is nondegenerate, i.e.,

$$\begin{pmatrix} y_1^{(r_1)} \\ y_2^{(r_2)} \\ \vdots \\ y_l^{(r_l)} \end{pmatrix} = \begin{bmatrix} c_1 A^{r_1} \\ c_2 A^{r_2} \\ \vdots \\ c_l A^{r_l} \end{bmatrix} x + H(r)u.$$

Importantly, the conditions in the Isidori definition may be inconsistent for a generic square linear stationary system [4].

Let us show that, under the above conditions, system (1) can be reduced to a special canonical form with zero dynamics separated out. More precisely, the system decomposes into two parts: the first describes the zero dynamics of the system, and its input is the output $y(t)$; and the second consists of the derivatives of various orders of the outputs y_i. We perform the transformation in several stages. First, we prove the following auxiliary lemma.

Lemma 1. *Suppose that system* (1) *is observable*, $\operatorname{rank} C = l$, $\operatorname{rank} B = l$, *and the conditions in the Isidori definition hold. Then, the rows* $c_1, c_1A, \ldots, c_1A^{r_1 - 1}, c_2, c_2A, \ldots, c_2A^{r_2 - 1}, c_3, \ldots, c_lA^{r_l - 1}$ *are linearly independent.*

Proof. Let us show that the set of vectors $c_1, c_1A, \ldots, c_lA^{r_l - 1}$ is linearly independent. Suppose that these vectors are linearly dependent. Then, there exist numbers $\gamma_1^1, \gamma_1^2, \ldots, \gamma_1^{r_1}, \gamma_2^1, \ldots, \gamma_l^{r_l}$, not all zero, such that

$$\sum_{i=1}^{l} \sum_{j=1}^{r_i} \gamma_i^j c_i A^{j-1} = 0. \qquad (6')$$

Let us multiply this identity by the matrix B on the right. Taking into account condition (i) in the definition of relative order, we obtain

$$\gamma_1^{r_1} c_1 A^{r_1 - 1} B + \gamma_2^{r_2} c_2 A^{r_2 - 1} B + \ldots + \gamma_l^{r_l} c_l A^{r_l - 1} B = 0.$$

Condition (ii) implies that the rows $c_i A^{r_i - 1} B$ are linearly independent; therefore, $\gamma_i^{r_i} = 0$ for all $i = 1, 2, \ldots, l$. Thus, (6') takes the form

$$\sum_{i=1}^{l} \sum_{j=1}^{r_l - 1} \gamma_i^j c_i A^{j-1} = 0.$$

Let us multiply this identity by AB on the right. Taking into account conditions (i) and (ii), we obtain $c_i^{r_i - 1} = 0$ for all $i = 1, 2, \ldots, l$. (If $r_j = 1$ for some j, then some terms may be missing from the linear combination of $c_i A^{r_i - 1} B$.) Continuing, we obtain $\gamma_i^j = 0$ for all i and j.

This completes the proof of the lemma.

Lemma 1 allows us to take the following variables for the new coordinates:

$$\begin{array}{lll} y_1^1 = c_1 x, & y_1^2 = c_2 x, & y_1^l = c_l x, \\ y_2^1 = c_1 A x, & y_2^2 = c_2 A x, & y_2^l = c_l A x, \\ \vdots & \vdots & \ldots \quad \vdots \\ y_{r_1}^1 = c_1 A^{r_1 - 1} x; & y_{r_2}^2 = c_2 A^{r_2 - 1} x; & y_{r_l}^l = c_l A^{r_l - 1} x. \end{array} \tag{7}$$

Let $|r| = r_1 + r_2 + \ldots + r_l$ denote the length of the vector r. Then, (7) determines $|r|$ coordinates. The remaining $n - |r|$ coordinates are chosen arbitrarily so that the transformation of coordinates be nondegenerate.

We set

$$\begin{pmatrix} y_1^1 \\ \vdots \\ y_{r_1}^1 \end{pmatrix} = \bar{y}_1, \ldots, \begin{pmatrix} y_1^l \\ \vdots \\ y_{r_l}^l \end{pmatrix} = \bar{y}_l$$

and denote the additional coordinates by $\bar{x}' \in \mathbb{R}^{n-|r|}$. In the new coordinates, the system takes the form

$$\begin{aligned} \dot{\bar{x}}' &= \bar{A}_{11}\bar{x}' + \sum_{i=1}^{l} \bar{A}_{12}^i \bar{y}_i + B_1 u, \\ \dot{y}_1^1 &= y_2^1, \\ \dot{y}_2^1 &= y_3^1, \\ &\vdots \\ \dot{y}_{r_1-1}^1 &= y_{r_1}^1; \\ \dot{y}_1^2 &= y_2^2, \\ &\vdots \\ \dot{y}_{r_2-1}^2 &= y_{r_2}^2; \\ &\vdots \\ \dot{y}_1^l &= y_2^l, \\ &\vdots \\ \dot{y}_{r_l-1}^l &= y_{r_l}^l; \\ \begin{pmatrix} \dot{y}_{r_1}^1 \\ \dot{y}_{r_2}^2 \\ \vdots \\ \dot{y}_{r_l}^l \end{pmatrix} &= \bar{A}_{21}\bar{x}' + \sum_{i=1}^{l} \bar{A}_{22}^i \bar{y}_i + H(r_1, r_2, \ldots, r_l) u, \end{aligned} \tag{8}$$

where $\bar{A}_{11}$, $\bar{A}_{21}$, $\bar{A}_{12}^i$, and $\bar{A}_{22}^i$ are constant matrices of the corresponding sizes, which are uniquely determined by the transformation of coordinates and the parameters of the system.

Note that the matrix $H(r) = H(r_1, r_2, \ldots, r_l)$ is nondegenerate; therefore, applying a nondegenerate transformation of the first $n - |r|$ coordinates

$$\tilde{x}' = \bar{x}' - B_1 H^{-1}(r) \begin{pmatrix} y_{r_1}^1 \\ \vdots \\ y_{r_l}^l \end{pmatrix},$$

we can render the first $n - |r|$ coordinates not depending explicitly on $u(t)$ at the second step. In the new coordinates, the first $n - |r|$ equations of the system take the form

$$\dot{\tilde{x}}' = \tilde{A}_{11}\tilde{x}' + \sum_{i=1}^{l} \tilde{A}_{12}^i \bar{y}_i, \tag{9}$$

where the matrices $\tilde{A}_{11}$ and $\tilde{A}_{12}^i$ are uniquely determined by the above change of coordinates.

In representation (9), the coordinates $\tilde{x}'$ depend explicitly not only on the outputs y_i but also on their derivatives (i.e., on the entire vectors $\bar{y}_i$). Let us show that these derivatives can be eliminated.

For this purpose, we eliminate the highest $(r_1 - 1)$th derivative $y_{r_1}^1 = y_1^{(r_1-1)}$ of the first output from representation (9). Let us write (9) in more detail:

$$\dot{\tilde{x}}' = \tilde{A}_{11}\tilde{x}' + \sum_{i=2}^{l} \tilde{A}_{12}^i \bar{y}_i + \sum_{j=1}^{r_1} (\tilde{A}_{12}^1)_j y_j^1, \tag{10}$$

where the $(\tilde{A}_{12}^1)_j$ are the columns of the matrix $\tilde{A}_{12}^1$. Taking into account the equality $\dot{y}_{r_1-1}^1 = y_{r_1}^1$, we make the change

$$\hat{x}' = \tilde{x}' - (\tilde{A}_{12}^1)_{r_1} y_{r_1-1}^1. \tag{11}$$

We obtain

$$\begin{aligned} \dot{\hat{x}}' &= \dot{\tilde{x}}' - (\tilde{A}_{12}^1)_{r_1} \dot{y}_{r_1-1}^1 \\ &= \tilde{A}_{11}\hat{x}' + \tilde{A}_{11} \cdot (\tilde{A}_{12}^1)_{r_1} y_{r_1-1}^1 + \sum_{i=2}^{l} \tilde{A}_{12}^i \bar{y}_i \\ &\quad + \sum_{j=1}^{r_1} (\tilde{A}_{12}^1)_j y_j^1 - (\tilde{A}_{12}^1)_{r_1} \cdot y_{r_1}^1 \end{aligned} \tag{12}$$

$$= \tilde{A}_{11}\hat{x}' + \sum_{i=2}^{l}(\tilde{A}_{12}^{i})\bar{y}_i + \sum_{j=1}^{r_1-1}(\hat{A}_{12}^{1})y_j^1.$$

Equation (12) differs from (10) in that it does not contain $y_{r_1}^1$ and in the column-coefficients of y_j^1 for $j = 1, 2, \ldots, r_1 - 1$. Thus, changing coordinates, we can get rid of $y_{r_1}^1$ in the initial $(n - |r|)$-dimensional part of the system.

At the next step, we get rid of $y_{r_1-1}^1$ in a similar way (with taking into account the equality $\dot{y}_{r_1-2}^1 = y_{r_1-1}^1$), then, of $y_{r_1-2}^1$, and so on, until the vector $\bar{y}_1$ contains only the coordinate $y_1^1 = y_1$, i.e., the first output of the system.

After this, by the same scheme, we successively eliminate the derivatives of the other outputs. Under the changes of coordinates specified above, in the remaining part, the systems of equations for y_j^i ($i = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, r_i - 1$) do not change; only the equations for higher derivatives of y_i, i.e., for the coordinates $y_{r_i}^i$, are affected.

We have proved the following theorem.

Theorem 1. *Suppose that, for a linear stationary square system of the form* (1), *the vector* $r = (r_1, r_2, \ldots, r_l)$ *of Isidori relative order is defined.*

Then, there exists a nondegenerate transformation reducing this system to the form

$$\begin{aligned} \dot{x}' &= A_{11}x' + A_{12}y, \\ \dot{y}_i^1 &= y_{i+1}^1, \quad i = 1, 2, \ldots, r_1 - 1; \\ \dot{y}_i^2 &= y_{i+1}^2, \quad i = 1, 2, \ldots, r_2 - 1; \\ &\ldots \\ \dot{y}_i^l &= y_{i+1}^l, \quad i = 1, 2, \ldots, r_l - 1; \\ \begin{pmatrix} \dot{y}_{r_1}^l \\ \vdots \\ \dot{y}_{r_l}^l \end{pmatrix} &= A_{21}x' + \sum_{i=1}^{l} A_{22}^i \bar{y}_i + H(r)u, \end{aligned} \tag{13}$$

which is a canonical representation of the system with zero dynamics separated out.

Corollary 1. *The zeroth dynamics of the system corresponds to the initial* $(n - |r|)$*-dimensional part of the system and is described by the equation*

$$\dot{x}' = A_{11}x'. \tag{14}$$

Corollary 2. *It is easy to show that, for system* (13), *the vector* $r = (r_1, r_2, \ldots, r_l)$ *satisfies the conditions in the definition of vector relative order in the sense of Isidori. Therefore, the reducibility of generic system* (1) *to the form* (13) *is a necessary and sufficient condition for the conditions in the definition of relative order to hold.*

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