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Morita equivalence based on contexts for various categories of modules over associative rings[1]

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Abstract

In this paper we consider the subcategories CMod-R ($M \in$ MOD-R s.t. $M \simeq \mathrm{Hom}_R(R,M)$) and DMod-$R$ ($M \in$ MOD-R s.t. $M \otimes_R R \simeq R$) of the category of all right R-modules, MOD-R, for an associative ring R, possibly without identity.

If R and S are associative rings and we have a Morita context between R and S with epimorphic pairings, it can be deduced from [6, 8] that the induced functors provide equivalences

CMod-$R \simeq$ CMod-S R-CMod $\simeq S$-CMod,
DMod-$R \simeq$ DMod-S R-DMod $\simeq S$-DMod.

We find hypotheses weaker than the surjectivity that let us prove also a converse of this result. As a consequence, we give an example of a ring R such that CMod-R is not equivalent to DMod-R.

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1. Introduction and preliminaries

In the following, all rings are associative rings but it is not assumed that they have an identity unless it is mentioned explicitly.

One of the main problems that appears when we want to study associative rings using categorical techniques is to choose an appropriate category of R-modules. The standard choice for rings with identity is the subcategory of MOD-R of the unitary R-modules, i.e. modules M such that $MR = M$. With more generality, a satisfactory Morita

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theory has been developed for rings with local units in [1] using the subcategory of unitary modules and for idempotent rings in [4] using the subcategory Mod-R of unitary modules that satisfy this additional condition

$$\forall m \in M, \quad mR = 0 \Rightarrow m = 0.$$

(In fact, this additional condition is always satisfied for unitary modules if R has local units or identity.) However, there are other rather natural (though possibly distinct) subcategories of MOD-R which arise in the nonunital case which could be considered as candidates for the "appropriate" category of modules. These are the categories CMod-R, DMod-R, and Mod-R described here.

Definition 1.1. Let R be an associative ring. We shall define the following three full subcategories of the category MOD-R:

(1) CMod-R is the category formed with the modules M such that $M \simeq \mathrm{Hom}_R(R,M)$ with the canonical homomorphism $\lambda_M : M \to \mathrm{Hom}_R(R,M)$ given by $\lambda_M(m)(r) = mr$ for all $m \in M$ and $r \in R$.

(2) DMod-R is the category formed with the modules M such that $M \simeq M \otimes_R R$ with the canonical homomorphism $\mu_M : M \otimes_R R \to M$ given by $\mu_M(m \otimes r) = mr$ for all $m \in M$ and $r \in R$.

(3) Mod-R is the category formed with the modules M such that $MR = M$ and for all $m \in M$ if $mR = 0$ then $m = 0$.

We shall give a direct proof of the fact that the categories CMod-R and DMod-R are equivalent to the one that has been used in the case of idempotent rings, and using the main theorem of this paper we will be able to prove that they are not equivalent in general.

Let R be a ring, and A the Dorroh's extension of R. This ring consists of the pairs $(r,z) \in R \times \mathbb{Z}$ with the sum defined componentwise and the product $(r,z)(r',z') = (rr' + rz' + r'z, zz')$. This is a ring with identity, $(0,1) = 1_A$, and R can be considered as a two-sided ideal of A if we identify the elements of R with the pairs $\{(r,0) \in A \colon r \in R\}$. The category of all right R-modules, MOD-R, is equivalent to the category of unitary right A-modules Mod-A. Also, the functors $- \otimes_R -$ and $\mathrm{Hom}_R(-,-)$ are, through the above equivalences, the same as $- \otimes_A -$ and $\mathrm{Hom}_A(-,-)$. All these facts are known or easily checked. For general properties of the Dorroh's extension of R, see [10, p. 5].

The categories that we are studying are full subcategories of MOD-R or R-MOD. In principle, every kernel, cokernel, exact sequence, etc., between R-modules will be considered in the category of unitary A-modules, therefore we will not worry about the existence of these objects. Properties of exactness of the functors $\mathrm{Hom}_R(P,-)$, $\mathrm{Hom}_R(-,Q)$, $P \otimes_R -$ and $- \otimes_R Q$ will be true because they are the same as the functors $\mathrm{Hom}_A(P,-)$, $\mathrm{Hom}_A(-,Q)$, $P \otimes_A -$ and $- \otimes_A Q$.

The category MOD-R = Mod-A is not the one that is used in order to study properties of the ring R. For instance, even in the case of a ring with identity S, the categories

Mod-S and MOD-S are rather different. In general terms, the modules of the category MOD-R that give problems are the modules M such that $MR = 0$.

Consider the class of modules in MOD-R such that $MR = 0$. Using the terminology of [9] this class is a pretorsion and pretorsion-free class. Therefore, we can define the associated preradical $\mathbf{t}$ in the following way:

$$\mathbf{t}(M) = \{m \in M:\ mR = 0\}.$$

This class of modules is the pretorsion class corresponding to the idempotent preradical $\mathbf{t}$. We can build the smallest radical larger than $\mathbf{t}$ as in [9, VI.1] in the following way: $\mathbf{t}_1 = \mathbf{t}$, if β is not a limit ordinal, then $\mathbf{t}_\beta$ is given by $\mathbf{t}_\beta(M)/\mathbf{t}_{\beta-1}(M) = \mathbf{t}(M/\mathbf{t}_{\beta-1}(M))$ and for a limit ordinal β, $\mathbf{t}_\beta = \sum_{\alpha<\beta} \mathbf{t}_\alpha$. For every module M, there exists an ordinal α such that $\mathbf{t}_\alpha(M) = \mathbf{t}_{\alpha+1}(M)$, then we define $\mathbf{T}(M) = \mathbf{t}_\alpha(M)$ for this α. This can be represented by $\mathbf{T} = \sum_\alpha \mathbf{t}_\alpha$, having in mind that, fixing a module M, this sum stabilizes for some ordinal.

The modules such that $\mathbf{T}(M) = M$ will be called *torsion modules* and the modules such that $\mathbf{T}(M) = 0$ (or equivalently $\mathbf{t}(M) = 0$) will be called *torsion-free modules.* The quotient category of MOD-R = Mod-A by this torsion theory will be denoted by CMod-R. This is the quotient category with respect to the R-adic topology in A. This shows that CMod-R is a Grothendieck category, although we will not use this fact.

The category CMod-R coincides with the category of modules M such that the canonical homomorphism λ_M: $M \to \mathrm{Hom}_R(R, M)$ $(\lambda_M(m)(r) = mr)$ is an isomorphism. This could be considered as the definition that we will use here because we will not use the properties of a quotient category. Dually, we will define DMod-R as the full subcategory of MOD-R formed with the modules M such that the canonical homomorphism μ: $M \otimes_R R \to M$ $(\mu(m \otimes r) = mr)$ is an isomorphism. The definitions of the converse are similar.

These categories have been considered with different notations in several papers, e.g. [6–8], but mainly as categories associated to the trace ideals of a Morita context between rings with identity. The definition of a Morita context for rings without identity is same as the one for rings with identity:

Definition 1.2. Let R and S be rings, ${}_SP_R$ and ${}_RQ_S$ bimodules and $\varphi: Q \otimes_S P \to R$, $\psi: P \otimes_R Q \to S$ bimodule homomorphisms. We say that $(R, S, P, Q, \varphi, \psi)$ is a Morita context if for all $p, p' \in P$ and $q, q' \in Q$, $\varphi(q \otimes p)q' = q\psi(p \otimes q')$ and $\psi(p \otimes q)p' = p\varphi(q \otimes p')$.

The two-sided ideals $\mathrm{Im}(\varphi)$ and $\mathrm{Im}(\psi)$ are called the trace ideals of the context.

The results given in [8, Theorem 3; 6, Theorem 2] determine the equivalences CMod-$\mathrm{Im}(\varphi) \simeq$ CMod-$\mathrm{Im}(\psi)$ and DMod-$\mathrm{Im}(\varphi) \simeq$ DMod-$\mathrm{Im}(\psi)$ and also on the other side. These results could be rewritten as follows: If we have a Morita context $(R, S, P, Q, \varphi, \psi)$ with φ and ψ epimorphisms, then CMod-$R \simeq$ CMod-S, DMod-$R \simeq$ DMod-S, R-CMod $\simeq$ S-CMod and R-DMod $\simeq$ S-DMod with the functors induced by the context. What we do here is to weaken the hypothesis "φ and ψ epimorphisms" in order to find also a converse of this result. Then we define a left-acceptable Morita context (Definition 3.6), such that all contexts with epimorphisms are left (and right) acceptable, and we obtain the characterization in Theorem 3.10.

If the ring R is idempotent, the class of modules M such that $MR=0$ is also closed under extensions. This is known as a TTF class. In this case, it can be deduced from [5, Proposition 1.15] that the categories CMod-R, DMod-R and the full subcategory of the modules such that $\mathbf{t}(M)=0$ and $MR=M$ are equivalent. In the case of idempotent rings, this category (in its different forms) has been chosen to develop a Morita theory; see, for example [1, 3, 4].

2. Some cases of equivalence

In this section we are going to study two different types of rings such that the categories CMod-R, DMod-R and Mod-R are equivalent.

Although this first case can be deduced from [5, Proposition 1.15] we shall give here a direct proof of the fact that the considered categories are equivalent if R is idempotent giving explicitly the functors in this case.

Definition 2.1. Let R be a ring. We shall use the following notations:

(1) $\mathbf{u}$ the functor that is defined over the objects of MOD-R as $\mathbf{u}(M)=MR$ and over the morphisms by the restriction.

(2) $j_M:\mathbf{u}(M)\to M$ the canonical inclusion.

(3) $\mathbf{t}^{\mathrm{opp}}$ the functor that is defined over the objects of MOD-R as $\mathbf{t}^{\mathrm{opp}}(M)=M/\mathbf{t}(M)$ and over the morphisms in the canonical way.

(4) $p_M:M\to M/\mathbf{t}(M)$ the canonical projection.

Lemma 2.2. *Let R be a ring and $M\in$ MOD-R. The morphisms*

$$\lambda_M:M\to \mathrm{Hom}_R(R,M),\quad \mu_M:M\otimes_R R\to M,$$
$$p_M:M\to \mathbf{t}^{\mathrm{opp}}(M),\quad j_M:\mathbf{u}(M)\to M.$$

define the natural transformations

$$\lambda:\mathrm{id}_{\mathsf{MOD}\text{-}R}\to \mathrm{Hom}_R(R,-),\quad \mu:-\otimes_R R\to \mathrm{id}_{\mathsf{MOD}\text{-}R},$$
$$p:\mathrm{id}_{\mathsf{MOD}\text{-}R}\to \mathbf{t}^{\mathrm{opp}},\quad j:\mathbf{u}\to \mathrm{id}_{\mathsf{MOD}\text{-}R}.$$

Proof. It is rather simple. □

Lemma 2.3. *Let $\gamma:R\otimes_R R\to R$ be the bimodule homomorphism defined as $\gamma(r\otimes s)=rs$. Then*

$$\mathrm{Ker}(\gamma)R=R\,\mathrm{Ker}(\gamma)=0.$$

Proof. Let $y\in \mathrm{Ker}(\gamma)$ and $x\in R$ then $yx=\gamma(y)\otimes x=0$ and $xy=x\otimes\gamma(y)=0$. □

Lemma 2.4. *Let R be a ring, $U,T,F \in$ MOD-R such that $UR=R$, $TR=0$, $\mathbf{t}(F)=0$ and $V \in R$-MOD such that $RV=0$. Then*

(1) $\mathrm{Hom}_R(U,T)=0$,

(2) $\mathrm{Hom}_R(T,F)=0$,

(3) $U \otimes_R V=0$.

Proof. (1) Let $f:U \to T$ and $u \in U=UR$. We can find elments $u_i \in U$ and $r_i \in R$ such that $u=\sum_i u_i r_i$ then $f(u)=\sum_i f(u_i)r_i=0$ because $f(u_i)r_i \in f(U)R \subseteq TR=0$.

(2) Let $f:T \to F$ and $t \in T$. For all $r \in R$, $f(t)r=f(tr)=0$; thus $f(t) \in \mathbf{t}(F)=0$.

(3) Let $u \in U$ and $v \in V$. We can find elements $u_i \in U$ and $r_i \in R$ such that $u=\sum_i u_i r_i$, then $u \otimes v=\sum_i u_i \otimes r_i v=0$ because $r_i v \in RV=0$. □

Proposition 2.5. *Let R be an idempotent ring and $M \in$ Mod-R. Then*

(1) $\mu_{M \otimes_R R}: M \otimes_R R \otimes_R R \to M \otimes_R R$ *is an isomorphism.*

(2) $\lambda_{\mathrm{Hom}_R(R,M)}: \mathrm{Hom}_R(R,M) \to \mathrm{Hom}_R(R,\mathrm{Hom}_R(R,M))$ *is an isomorphism.*

Proof. (1) $\mu_{M \otimes_R R} \simeq M \otimes_R \gamma$ with $\gamma: R \otimes_R R \to R$ the canonical homomorphism. As R is idempotent, γ is an epimorphism and $M \otimes_R \gamma$ is an epimorphism. Using the fact that $R\,\mathrm{Ker}(\gamma)=0$, we deduce from Lemma 2.4 that $M \otimes_R \mathrm{Ker}(\gamma)=0$ and the exactness of the sequence

$$M \otimes_R \mathrm{Ker}(\gamma) \to M \otimes_R R \otimes_R R \to M \otimes_R R \to 0,$$

completes the proof.

(2) Using the canonical isomorphism

$$\mathrm{Hom}_R(R,\mathrm{Hom}_R(R,M)) \simeq \mathrm{Hom}_R(R \otimes_R R, M),$$

it is easy to check that $\lambda_{\mathrm{Hom}_R(R,M)}$ is an isomorphism if and only if $\mathrm{Hom}_R(\gamma,M)$ is an isomorphism. As γ is an epimorphism, $\mathrm{Hom}_R(\gamma,M)$ is a monomorphism and using the exactness of the sequence

$$0 \to \mathrm{Hom}_R(R,M) \to \mathrm{Hom}_R(R \otimes_R R, M) \to \mathrm{Hom}_R(\mathrm{Ker}(\gamma), M)$$

and the fact that $\mathrm{Hom}_R(\mathrm{Ker}(\gamma),M)=0$ (because $\mathrm{Ker}(\gamma)R=0$ and $\mathbf{t}(M)=0$) we deduce from Lemma 2.4 that $\mathrm{Hom}_R(\gamma,M)$ is an isomorphism. □

Proposition 2.6. *Let R be an idempotent ring. Then*

(1) *for all $M \in$ CMod-R, $\mathbf{u}(M) \in$ Mod-R,*

(2) *for all $M \in$ DMod-R, $\mathbf{t}^{\mathrm{opp}}(M) \in$ Mod-R.*

Proof. (1) As $\mathbf{t}(M)=\mathrm{Ker}(\lambda_M)=0$ we know that $\mathbf{t}(\mathbf{u}(M)) \subseteq \mathbf{t}(M)=0$. On the other hand, using the fact that $R^2=R$ we deduce that $\mathbf{u}(M)R=MR^2=MR=\mathbf{u}(M)$.

(2) As $MR=\mathrm{Im}(\mu_M)=M$ we know that $(M/\mathbf{t}(M))R=M/\mathbf{t}(M)$. On the other hand, if $(m+\mathbf{t}(M))R=0$ then $mR \subseteq \mathbf{t}(M)$ and $mR^2=0$. Using the fact that $R^2=R$ we deduce that $m \in \mathbf{t}(M)$ and therefore $m+\mathbf{t}(M)=0$. □

Proposition 2.7. *Let R be an idempotent ring. Using the previous propositions we have functors*

$$\mathbf{u}\colon \mathsf{CMod}\text{-}R \to \mathsf{Mod}\text{-}R, \quad \mathrm{Hom}_R(R,-)\colon \mathsf{Mod}\text{-}R \to \mathsf{CMod}\text{-}R,$$

$$\mathbf{t}^{\mathrm{opp}}\colon \mathsf{DMod}\text{-}R \to \mathsf{Mod}\text{-}R, \quad -\otimes_R R\colon \mathsf{Mod}\text{-}R \to \mathsf{DMod}\text{-}R.$$

These functors are equivalences with the natural transformations given by
(1) $\lambda_X^{-1} \circ \mathrm{Hom}_R(R, j_X)\colon \mathrm{Hom}_R(R,\mathbf{u}(X)) \to X$ *for all* $X \in \mathsf{CMod}\text{-}R$.
(2) $\mathbf{u}(\lambda_M)\colon M \to \mathbf{u}(\mathrm{Hom}_R(R,M))$ *for all* $M \in \mathsf{Mod}\text{-}R$.
(3) $\mathbf{t}^{\mathrm{opp}}(\mu_M)\colon \mathbf{t}^{\mathrm{opp}}(M \otimes_R R) \to M$ *for all* $M \in \mathsf{Mod}\text{-}R$.
(4) $(p_Y \otimes_R R) \circ \mu_Y\colon Y \to \mathbf{t}^{\mathrm{opp}}(Y) \otimes_R R$ *for all* $Y \in \mathsf{DMod}\text{-}R$.

Proof. (1) As λ_X is an isomorphism, we only have to check that $\mathrm{Hom}_R(R,j_X)$ is an isomorphism. The morphism j_X is a monomorphism, therefore $\mathrm{Hom}_R(R,j_X)$ is a monomorphism. Using the exactness of the sequence

$$0 \to \mathrm{Hom}_R(R,\mathbf{u}(X)) \to \mathrm{Hom}_R(R,X) \to \mathrm{Hom}_R(R,\mathrm{Coker}(j_X))$$

and the fact that $R^2 = R$ and $\mathrm{Coker}(j_X)R = 0$ we deduce that $\mathrm{Hom}_R(R,\mathrm{Coker}(j_X)) = 0$ and $\mathrm{Hom}_R(R,j_X)$ is an isomorphism.

(2) $\mathrm{Ker}(\mathbf{u}(\lambda_M)) = \mathbf{t}(\mathbf{u}(M)) \subseteq \mathbf{t}(M) = 0$. This proves that $\mathbf{u}(\lambda_M)$ is a monomorphism. To check that it is an epimorphism, let $\sum_i f_i r_i \in \mathrm{Hom}_R(R,M)R$ with $f_i\colon R \to M$ and $r_i \in R$. It is easy to prove that $\sum_i f_i r_i = \mathbf{u}(\lambda_M)(\sum_i f_i(r_i))$.

(3) As $\mathrm{Ker}(\mu_M)R = 0$, we deduce that $\mathrm{Ker}(\mu_M) \subseteq \mathbf{t}(M \otimes_R R)$ and therefore $\mathbf{t}^{\mathrm{opp}}(\mu_M)$ is a monomorphism. Let $\sum_i m_i r_i \in M = MR$, then $\mathbf{t}^{\mathrm{opp}}(\sum_i m_i \otimes r_i + \mathbf{t}(M \otimes_R R)) = \sum_i m_i r_i$ and this proves the surjectivity of $\mathbf{t}^{\mathrm{opp}}(\mu_M)$.

(4) As μ_Y is an isomorphism, we only have to check that $p_Y \otimes_R R$ is an isomorphism. Using the exactness of the sequence

$$\mathbf{t}(M) \otimes_R R \to Y \otimes_R R \to \mathbf{t}^{\mathrm{opp}}(M) \otimes_R R \to 0$$

and that $\mathbf{t}(M) \otimes_R R = 0$ (because $\mathbf{t}(M)R = 0$ and $R^2 = R$) we deduce from Lemma 2.4 that $p_Y \otimes_R R$ is an isomorphism. □

We now give a second type of ring for which the categories $\mathsf{CMod}\text{-}R$, $\mathsf{DMod}\text{-}R$, and $\mathsf{Mod}\text{-}R$ are equivalent; in fact, in this case these three subcategories of $\mathsf{MOD}\text{-}R$ are equal.

Definition 2.8. Let R be a ring and $g_R \in R$. We shall say that g_R is a central generator of R if
(1) g_R commutes with the elements of R,
(2) $g_R R + g_R \mathbb{Z} = R$,
(3) for all $r \in R$, $r g_R = 0 \Rightarrow r = 0$.

All rings with identity are rings with a central generator taking g_R as the identity element. All the ideals in a p.i.d. are also rings with a central generator.

Proposition 2.9. *Let R be a ring with a central generator g_R. The following conditions on $M \in$ MOD-R are equivalent*:

(1) $M \in$ CMod-R.

(2) $M \in$ Mod-R.

(3) $M \in$ DMod-R.

Proof. Let A be the Dorroh's extension of R. A module M satisfies $MR = M$ if and only if $M = MR = MAg_R = Mg_R$. For any module M,

$$\mathbf{t}(M) = \{m \in M:\ mR = 0\} = \{m \in M:\ mg_RA = 0\}$$
$$= \{m \in M:\ mg_R = 0\}.$$

((1)⇒(2)): Let $M \in$ CMod-R. Clearly, M is torsion free because $\mathbf{t}(M) = \operatorname{Ker}(\lambda_M) = 0$. In order to prove that $M = MR$, let $m \in M$ and consider the homomorphism $f : R \to M$ defined by $f(g_Ra) = ma$ for all $a \in A$. This definition is good because if for some $r \in R$, $r = g_Ra = g_Ra'$ then $g_R(a - a') = 0 = (a - a')g_R \Rightarrow a = a'$. Using the fact that $M \in$ CMod-R we can find an element $m' \in M$ such that $f(r) = m'r$ for all $r \in R$, then $m = f(g_R) = m'g_R \in MR$.

((2)⇒(3)): Let $M \in$ Mod-R. All the elements in $M \otimes_R R$ can be written as $m \otimes g_R$ for some $m \in M$. If $m \otimes g_R \in \operatorname{Ker}(\mu_M)$, then $mg_R = 0$ and $m \in \mathbf{t}(M) = 0$. This proves that $\operatorname{Ker}(\mu_M) = 0$. The surjectivity of μ_R is clear because $MR = M$.

((3)⇒(1)): Let $M \in$ DMod-R and $m \in \operatorname{Ker}(\lambda_M)$, then $mg_R = 0$ and $m \otimes g_R = 0$. Using the result given in [10, p. 97] we can find elements $m_k \in M$ and $a_k \in A$ for $k = 1, \ldots, n$ such that $a_kg_R = 0$ for all k and $m = \sum_{k=1}^{n} m_ka_k$. The elements $m_k \in M = MR$; therefore, we can find $m'_k \in M$ such that $m_k = m'_kg_R$ for all $k \in \{1, \ldots, n\}$. Thus,

$$m = \sum_{k=1}^{n} m_ka_k = \sum_{k=1}^{n} m'_ka_kg_R = 0.$$

In order to prove the surjectivity of λ_M let $f : R \to M$ be a homomorphism. The element $f(g_R) \in M = MR$, therefore there exists an $m \in M$ such that $f(g_R) = mg_R$. What we are going to prove is that $f = \lambda_M(m)$. Let $r \in R$, $r = g_Ra$ for some $a \in A$, then $f(r) = f(g_R)a = mg_Ra = mr = \lambda_M(m)(r)$. □

3. Contexts and equivalences

First of all we are going to build a module in the category R-DMod. This construction follows the steps of that in [2, 28.1], but we shall prove further properties of the module that is built there.

Let R be a ring, A the Dorroh's extension of R, $(r_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ such that $r_1r_2 \cdots r_n \neq 0$ for all $n \in \mathbb{N}$. Let $F = A^{(\mathbb{N})}$ the free left A-module over the set $\mathbb{N}$. We shall denote for all $n \in \mathbb{N}$, v_n the element in F that has 1_A in the nth component and 0 elsewhere, $u_n = v_n - r_{n+1}v_{n+1}$, G will be the module $\sum_{n \in \mathbb{N}} Au_n$ and $M = F/G$. We are going to prove that $M \in R$-DMod. With the same proof of [2, 28.1] it is possible to see that for all $n \in \mathbb{N}$ and $a \in A$, $av_n + G = 0$ if and only if there exists $k \geq n$ such that $ar_{n+1} \cdots r_k = 0$.

This proves that the module M is not 0 because the element $v_0 + G$ can never be 0. If $v_0 + G = 0$ we would find a $k \in \mathbb{N}$ such that $1_A r_1 \cdots r_k = 0$ and this is not possible because $r_1 \cdots r_k \neq 0$. In order to prove that $M \in R$-DMod consider the canonical morphism $\mu_M : R \otimes_R M \to M$. This morphism is an epimorphism because for all $n \in \mathbb{N}$, $v_n + G = r_{n+1} v_{n+1} + G$. The elements in $R \otimes_R M$ can be written like $r \otimes v_n + G$ with $r \in R$ and $n \in \mathbb{N}$. If $rv_n + G = \mu(r \otimes v_n + G) = 0$ then there exists $k \geq n$ such that $rr_{n+1} \cdots r_k = 0$ and therefore $r \otimes v_n + G = rr_{n+1} \ldots r_k \otimes v_k + G = 0$.

Lemma 3.1. *Let $M \in$ MOD-R. Then*

$$\mathbf{T}(M) = \{m \in M \colon \forall (r_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}\ \exists n_0 \in \mathbb{N}\ s.t.\ mr_1 \cdots r_{n_0} = 0\}.$$

Proof. Assume first that $m \in \mathbf{T}(M)$. As we observed earlier, there is an ordinal α such that $\mathbf{T}(M) = \mathbf{t}_\alpha(M)$, so that $\mathbf{t}_\alpha(M) = \mathbf{t}_{\alpha+1}(M)$. Let $(r_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ and $m \in \mathbf{T}(M)$ such that $mr_1 \cdots r_n \neq 0$ for all $n \in \mathbb{N}$. We know that $m \in \mathbf{T}(M) = \mathbf{t}_\alpha(M)$, therefore, we can find a smallest ordinal γ_0 such that $m \in \mathbf{t}_{\gamma_0}(M)$. For $i = 1, 2, \ldots, n$ we now define a nonzero ordinal γ_i, as the first ordinal such that $mr_1 \cdots r_i \in \mathbf{t}_{\gamma_i}(M)$.

By our hypothesis that the given sequence does not annihilate m, we see that γ_i cannot be 0. Also, by the construction of the $\mathbf{t}_\alpha$, each γ_i is a successor ordinal (if it were a limit ordinal, then a contradiction would arise from the fact that $mr_1 \cdots r_i \in \mathbf{t}_{\gamma_i}(M) = \sum_{\beta < \gamma_i} \mathbf{t}_\beta(M)$, but $mr_1 \ldots r_i \notin \mathbf{t}_\beta(M)$ for $\beta < \gamma_i$).

In order to compare now γ_i and γ_{i+1}, suppose $\gamma_i = \beta + 1$. Clearly, $\gamma_{i+1} \leq \beta + 1$. But we have $mr_1 \cdots r_i \in \mathbf{t}_{\beta+1}(M)$. By the construction of the $\mathbf{t}_\alpha$, we infer that the class of $mr_1 \cdots r_i$ modulo $\mathbf{t}_\beta(M)$ is annihilated by R, that is $mr_1 \cdots r_i R \subseteq \mathbf{t}_\beta(M)$. In particular, $mr_1 \cdots r_{i+1} \in \mathbf{t}_\beta(M)$. This implies that $\gamma_{i+1} \leq \beta < \gamma_i$. This shows that the decreasing sequence of the ordinals γ_i is strictly decreasing. But any set of ordinals has a smallest element, which contradicts the existence of the sequence of the γ_i. This is the contradiction we were looking for.

We turn now to the converse part of the proof. Assume that $m \notin \mathbf{T}(M) = \mathbf{t}_\alpha(M)$. As $\mathbf{t}_\alpha(M) = \mathbf{t}_{\alpha+1}(M)$, then mR is not contained in $\mathbf{t}_\alpha(M)$ and we can find $r_1 \in R$ such that $mr_1 \notin \mathbf{t}_\alpha(M) = \mathbf{T}(M)$. In the same way, once we have obtained that $mr_1 \cdots r_k \notin \mathbf{t}_\alpha(M)$, we infer that $mr_1 \cdots r_k R$ is not contained in $\mathbf{t}_\alpha(M)$. So we find r_{k+1} such that $mr_1 \cdots r_k r_{k+1} \notin \mathbf{t}_\alpha(M)$. In particular, each of these products is nonzero. □

Definition 3.2. Let $M \in$ MOD-R and L a R-submodule of M. As in [9, Section IX.4] we define L^c as the biggest submodule of M such that L^c/L is torsion.

Lemma 3.3. *Let Z be an abelian group, $W \in R$-MOD such that $RW = W$, $M \in$ MOD-R and L_0 a subset of M. Let $h : M \otimes_R W \to Z$ be an abelian group homomorphism such that $h(l \otimes w) = 0$ for all $l \in L_0$ and $w \in W$, then if we denote L the right R-submodule of M generated by L_0, for all $l' \in L^c$ and all $w \in W$, $h(l' \otimes w) = 0$.*

Proof. Clearly, $h(l \otimes w) = 0$ for all $l \in L$ because L is the smallest submodule that contains L_0. Suppose some $l' \in L^c$ and some $w \in W$ that $h(l' \otimes w) \neq 0$. We claim that then there exist sequences $(r_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ and $(w_n)_{n \in \mathbb{N}} \in W^{\mathbb{N}}$ such that $h(l' r_1 \cdots r_n \otimes w_n) \neq 0$. Once this is proved we get a contradiction because using Lemma 3.1, for some $n_0 \in \mathbb{N}$, $l' r_1 \cdots r_{n_0} \in L$ and then $h(l' r_1 \cdots r_{n_0} \otimes w_{n_0}) = 0$. Let us prove the claim.

For $n = 0$ we have $w_0 = w$, suppose we have $r_1, \ldots, r_n \in R$ and $w_n \in W$ such that $h(l' r_1 \cdots r_n \otimes w_n) \neq 0$. Using the fact that $RW = W$ we can find elements $w_i' \in W$ and $r_i' \in R$ for $i = 1, \ldots, k$ such that $w_n = \sum_{i=1}^k r_i' w_i'$. Then $0 \neq h(l' r_1 \cdots r_n \otimes w_n) = \sum_{i=1}^k h(l' r_1 \cdots r_n r_i' \otimes w_i')$ and therefore, for some $i_0 \in \{1, \ldots, k\}$, $h(l' r_1 \cdots r_n r_{i_0}' \otimes w_{i_0}') \neq 0$. If we define $r_{n+1} = r_{i_0}'$ and $w_{n+1} = w_{i_0}'$ we can continue the induction process. □

Proposition 3.4. *Let $M \subseteq N$ modules in MOD-R with $M \in$ CMod-R and N torsion-free. Then N/M is torsion-free.*

Proof. Let $n \in N$ such that $nR \subseteq M$, then we define $f : R \to M$ such that $f(r) = nr$. Using the fact that $M \in$ CMod-R we can find $m \in M$ such that $mr = f(r) = nr$ for all $r \in R$ and then $n = m \in M$ because N is torsion-free. □

Proposition 3.5. *Let $M \subseteq N$ modules in R-MOD with $N/M \in R$-DMod and $RN = N$. Then $RM = M$.*

Proof. Let $m \in M$, as $m \in N$ we can find elements $n_i \in N$ and $r_i \in R$ such that $m = \sum_{i=1}^k r_i n_i$. Consider the element $\sum_{i=1}^k r_i \otimes (n_i + M) \in R \otimes_R N/M$. This element satisfies $\sum_{i=1}^k r_i(n_i + M) = 0$ and using the fact that $N/M \in R$-DMod, we deduce $\sum_{i=1}^k r_i \otimes (n_i + M) = 0$. Then using the exactness of the sequence $R \otimes_R M \to R \otimes_R N \to R \otimes_R N/M \to 0$ we can find elements $r_j' \in R$ and $m_j' \in M$ such that $\sum_{i=1}^k r_i \otimes n_i = \sum_{j=1}^t r_j' \otimes m_j'$, therefore $m = \sum_{j=1}^t r_j' m_j' \in RM$ and then $RM = M$. □

Definition 3.6. Let $(R, S, P, Q, \varphi, \psi)$ be a Morita context. We shall say that this context is left acceptable if and only if

(1) $\forall (r_n)_{n \in \mathbb{N}} \in R^{\mathbb{N}}$ $\exists n_0 \in \mathbb{N}$ *such that* $r_1 \cdots r_{n_0} \in \operatorname{Im}(\varphi)$,

(2) $\forall (s_m)_{m \in \mathbb{N}} \in S^{\mathbb{N}}$ $\exists m_0 \in \mathbb{N}$ *such that* $s_1 \cdots s_{m_0} \in \operatorname{Im}(\psi)$.

Remark 3.7. The definition of right acceptable Morita contexts is the one that we obtain if we use the opposite rings and bimodules.

These definitions include the case in which φ and ψ are epimorphisms, but in the case of idempotent rings it is possible to prove that this is not a proper extension.

Proposition 3.8. *Let R and S be idempotent rings and $(R, S, P, Q, \varphi, \psi)$ a Morita context. The following conditions are equivalent*:

(1) *$(R, S, P, Q, \varphi, \psi)$ is left acceptable.*

(2) *$(R, S, P, Q, \varphi, \psi)$ is right acceptable.*

(3) *φ and ψ are surjective.*

Proof. Clearly, (3) implies conditions (1) and (2). In fact, we only have to prove ((1)⇒(3)) or ((2)⇒(3)) because of the symmetry of the third condition.

((1)⇒(3)) Let $\bar{r} \in R \backslash \mathrm{Im}(\varphi)$. Because of the idempotence of R, we can find elements r_i^1 and $\bar{r}_i^1$ such that $\bar{r} = \sum_i r_i^1 \bar{r}_i^1$. As $\sum_i r_i^1 \bar{r}_i^1 \notin \mathrm{Im}(\varphi)$ there exists i_1 such that $r_{i_1}^1 \bar{r}_{i_1}^1 \notin \mathrm{Im}(\varphi)$.

For the element $\bar{r}_{i_1}^1$ we can find elements r_i^2 and $\bar{r}_i^2$ in R such that $\bar{r}_{i_1}^1 = \sum_i r_i^2 \bar{r}_i^2$, then for some i_2, $r_{i_1}^1 r_{i_2}^2 \bar{r}_{i_2}^2 \notin \mathrm{Im}(\varphi)$. Following this argument we can find a sequence $(r_{i_k}^k) \in R^{\mathbb{N}}$ and $\bar{r}_{i_k}^k$ such that

$$r_{i_1}^1 r_{i_2}^2 \cdots r_{i_k}^k \bar{r}_{i_k}^k \notin \mathrm{Im}(\varphi) \quad \forall k \in \mathbb{N}$$

and this is a contradiction with the fact that for some $k \in \mathbb{N}$, $r_{i_1}^1 r_{i_2}^2 \cdots r_{i_k}^k \in \mathrm{Im}(\varphi)$.

This proves the surjectivity of φ. The proof for ψ is similar. □

If the rings are not idempotent, this relation need not be true; for instance, for any ring R, the context given by (R,R,R,R,γ,γ) with the $\gamma : R \otimes_R R \to R$ given in Lemma 2.3 is always left and right acceptable, but γ is surjective if and only if R is idempotent.

Although we do not need to have surjectivity, the relation between left-acceptable contexts and right-acceptable ones appears also in commutative rings and also in rings with a central generator.

Proposition 3.9. *Let R and S be general associative rings and let (R,S,P,Q,φ,ψ) be a left-acceptable Morita context. Then*:

(1) *$M_R \in$ MOD-R is torsion-free if and only if for all $m \in M$, $m\,\mathrm{Im}(\varphi) = 0$ implies $m = 0$.*

(2) *${}_RM \in R$-MOD satisfies $RM = M$ if and only if $\mathrm{Im}(\varphi)M = M$.*

Proof. Let $M_R \in$ MOD-R. Suppose M is not torsion-free, then we can find $m \in \mathbf{t}(M) \backslash 0$ with $mR = 0$, and so $m\,\mathrm{Im}(\varphi) = 0$. On the other hand, suppose M torsion-free and $m \in M$, $m \neq 0$, using Lemma 3.1 we can find a sequence $(r_n)_{n \in \mathbb{N}}$ such that $mr_1 \cdots r_n \neq 0$ for all $n \in \mathbb{N}$. As the context is left acceptable we can find a $n_0 \in \mathbb{N}$ such that $r_1 \cdots r_{n_0} \in \mathrm{Im}(\varphi)$ and then $m\,\mathrm{Im}(\varphi) \neq 0$.

Let ${}_RM \in R$-MOD such that $\mathrm{Im}(\varphi)M = M$ then $M \supseteq RM \supseteq \mathrm{Im}(\varphi)M = M$. On the other hand, suppose $RM = M$ and let $h : R \otimes_R M \to M/\mathrm{Im}(\varphi)M$ be the homomorphism defined as $h(r \otimes m) = rm + \mathrm{Im}(\varphi)M$. Clearly, $h(\varphi(q \otimes p) \otimes m) = 0$ for all $q \in Q$, $p \in P$ and $m \in M$. Using Lemma 3.3 and the fact that the context is left acceptable, we deduce that $h(r \otimes m) = 0$ for all $r \in R$ and $m \in M$ and then $RM/\mathrm{Im}(\varphi)M = 0$ and $\mathrm{Im}(\varphi)M = RM = M$. □

Theorem 3.10. *Let (R,S,P,Q,φ,ψ) be a Morita context. The following conditions are equivalent*:

(1) *$\mathrm{Hom}_R(P_R, -)$ and $\mathrm{Hom}_S(Q_S, -)$ are inverse category equivalences with the transformations given by the context, between the categories CMod-R and CMod-S.*

(2) *$P\otimes_R-$ and $Q\otimes_S-$ are inverse category equivalences with the transformations given by the context, between the categories R-DMod and S-DMod.*

(3) *The context (R,S,P,Q,φ,ψ) is left acceptable.*

Proof. ((3)⇒(1)): The pairings induce the following natural transformations:

$$\Phi:\mathrm{id}_{\mathsf{MOD}\text{-}R}\to \mathrm{Hom}_S(Q,\mathrm{Hom}_R(P,-))\simeq \mathrm{Hom}_R(Q\otimes_S P,-),$$

$$\Psi:\mathrm{id}_{\mathsf{MOD}\text{-}S}\to \mathrm{Hom}_R(P,\mathrm{Hom}_S(Q,-))\simeq \mathrm{Hom}_S(P\otimes_R Q,-),$$

given by $\Phi_X(x)(q)(p)=x\varphi(q\otimes p)$ and $\Psi_Y(y)(p)(q)=y\psi(p\otimes q)$.

Let $X\in\mathsf{MOD}\text{-}R$. Then $\mathrm{Ker}(\Phi_X)=\{x\in X\mid \forall p\in P,\ \forall q\in Q,\ x\varphi(q\otimes p)=0\}$. Using Proposition 3.9 we deduce that $\mathbf{t}(X)=0$ if and only if Φ_X is injective.

Consider the canonical homomorphism $\lambda_X:X\to\mathrm{Hom}_R(R,X)$ $(\lambda_X(x)(r)=xr))$. What we have to prove is that $X\in\mathsf{CMod}\text{-}R$ (i.e. λ_X is a isomorphism) if and only if Φ_X is an isomorphism. As $\mathrm{Ker}(\lambda_X)=\mathbf{t}(X)$ we have proved that λ_X is injective iff Φ_X is injective.

Suppose that X is torsion-free, we claim that $\mathrm{Hom}_R(R,X)$ and $\mathrm{Hom}_R(Q\otimes P,X)$ are torsion-free. If $f:R\to X$ satisfies $fR=0$ then for all $r\in R$ and $r'\in R$ we have $0=fr(r')=f(rr')=f(r)r'$, then $f(r)R=0$ and using the fact that X is torsion-free, we deduce that $f(r)=0$ for all $r\in R$ and then $f=0$. Let now $f:Q\otimes_S P\to X$ such that $fR=0$ and let $p,p'\in P$ and $q,q'\in Q$, $0=f\varphi(q\otimes p)(q'\otimes p')=f(\varphi(q\otimes p)q'\otimes p')=f(q\psi(p\otimes q')\otimes p')=f(q\otimes p\varphi(q'\otimes p'))=f(q\otimes p)\varphi(q'\otimes p')$. Then $f(q\otimes p)\mathrm{Im}(\varphi)=0$ and using Proposition 3.9 and the fact that X is torsion-free we deduce that $f(q\otimes p)=0$ for all $p\in P$ and $q\in Q$, therefore $f=0$.

Suppose that Φ_X is an isomorphism, then, X is torsion-free and λ_X is a monomorphism. As $\Phi_X=\mathrm{Hom}_R(\varphi,X)\circ\lambda_X$ we deduce that $\mathrm{Coker}(\lambda_X)$ is a direct summand of $\mathrm{Hom}_R(R,X)$ and therefore, it is torsion-free, but $\mathrm{Coker}(\lambda_X)R=0$, then $\mathrm{Coker}(\lambda_X)=0$ and λ_X is an isomorphism. On the other hand, suppose λ_X is an isomorphism. Then $\mathbf{t}(X)=0$ and Φ_X is a monomorphism. Using Proposition 3.4 we deduce that $\mathrm{Coker}(\Phi_X)$ is torsion-free, but $\mathrm{Coker}(\Phi_X)\mathrm{Im}(\varphi)=0$, then using Proposition 3.9 we conclude $\mathrm{Coker}(\Phi_X)=0$ and therefore Φ_X is an isomorphism.

In a similar way, it is possible to prove that $Y\in\mathsf{CMod}\text{-}S$ if and only if Ψ_Y is an isomorphism. Using these facts it is not difficult to check that $\mathrm{Hom}_R(P,-)$ and $\mathrm{Hom}_S(Q,-)$ are inverse category equivalences between $\mathsf{CMod}\text{-}R$ and $\mathsf{CMod}\text{-}S$.

((1)⇒(3)): Suppose that there exists a sequence $(r_n)_{n\in\mathbb{N}}\in R^{\mathbb{N}}$ such that for all $n\in\mathbb{N}$, $r_1\cdots r_n\notin\mathrm{Im}(\varphi)$, we are going to build a module X in $\mathsf{CMod}\text{-}R$ such that Φ_X is not an isomorphism. As $K/\mathrm{Im}(\varphi):=\mathbf{T}(R/\mathrm{Im}(\varphi))\neq R/\mathrm{Im}(\varphi)$, then the module $R/K\neq 0$ is torsion-free. If we apply the localization functor, $X:=\mathbf{a}(R/K)$ is a nonzero module in $\mathsf{CMod}\text{-}R$ and the elements $r+K\in R/K\subseteq X$ have the property $(r+K)\mathrm{Im}(\varphi)=0$ because $r\,\mathrm{Im}(\varphi)\subseteq K$, then $R/K\subseteq\mathrm{Ker}(\Phi_X)$ and therefore Φ_X is not an isomorphism.

((3)⇒(2)): The pairings induce the following natural transformations:

$$\Delta:Q\otimes_S P\otimes_R-\to\mathrm{id}_{R\text{-}\mathsf{MOD}},\qquad \Lambda:P\otimes_R Q\otimes_S-\to\mathrm{id}_{S\text{-}\mathsf{MOD}}$$

given by $\Delta_X(q\otimes p\otimes x)=\varphi(q\otimes p)x$ and $\Lambda_Y(p\otimes q\otimes y)=\psi(p\otimes q)y$. Let $X\in R$-MOD. Then $\mathrm{Im}(\Delta_X)=\mathrm{Im}(\varphi)X$ and using Proposition 3.9 we deduce that $RX=X$ if and only if Δ_X is an epimorphism. We have proved that Δ_X is an epimorphism if and only if the canonical morphism $\mu_X : R\otimes X\to X$ is an epimorphism.

If $X=RX$ then $R(R\otimes_R X)=R\otimes_R X$, and using Proposition 3.9 $\mathrm{Im}(\varphi)X=X$, therefore $R(Q\otimes_S P\otimes_R X)=Q\otimes_S P\otimes_R X$ because if $\sum q_i\otimes p_i\otimes x_i\in Q\otimes_S P\otimes_R X$, for the elements x_i we can find elements $q'_{ij}\in Q$, $p'_{ij}\in P$ and $x'_{ij}\in X$ such that $x_i=\sum_j \varphi(q'_{ij}\otimes p'_{ij})x'_{ij}$ and then $\sum_i q_i\otimes p_i\otimes x_i=\sum_{ij} q_i\otimes p_i\otimes \varphi(q'_{ij}\otimes p'_{ij})x'_{ij}=\sum_{ij} q_i\otimes \psi(p_i\otimes q'_{ij})p'_{ij}\otimes x'_{ij}=\sum_{ij}\varphi(q_i\otimes p_i)q'_{ij}\otimes p'_{ij}\otimes x'_{ij}\in R(Q\otimes_S P\otimes_R X)$.

We have to prove that Δ_X is an isomorphism if and only if μ_X is an isomorphism. Suppose that Δ_X is an isomorphism, then $RX=X$ and therefore μ_X is an epimorphism. As $\mu_X\circ(\varphi\otimes X)=\Delta_X$, then $\mathrm{Ker}(\mu_X)$ is a direct summand of $R\otimes_R X$ and therefore $R\,\mathrm{Ker}(\mu_X)=\mathrm{Ker}(\mu_X)$. But it is a simple computation to check that $R\,\mathrm{Ker}(\mu_X)=0$, therefore μ_X is an isomorphism. On the other hand, suppose μ_X is an isomorphism, then using Proposition 3.5 we deduce that $R\,\mathrm{Ker}(\Delta_X)=\mathrm{Ker}(\Delta_X)$ and then $\mathrm{Im}(\varphi)\mathrm{Ker}(\Delta_X)=\mathrm{Ker}(\Delta_X)$, but it is not difficult to see that $\mathrm{Im}(\varphi)\mathrm{Ker}(\Delta_X)=0$ and therefore Δ_X is an isomorphism.

In a similar way, it is possible to deduce that Λ_Y is an isomorphism if and only if $Y\in S$-DMod. With these results it is not difficult to prove that $P\otimes_R -$ and $Q\otimes_S -$ are inverse category equivalences between R-DMod and S-DMod.

((2)$\Rightarrow$(3)): Suppose Δ_X is an isomorphism for all $X\in R$-DMod. We claim first that for all $p\in P$ and $q\in Q$ and all sequence $(r_n)_{n\in\mathbb{N}}\in R^{\mathbb{N}}$ there exists an $m\in\mathbb{N}$ such that $\varphi(q\otimes p)r_1\ldots r_m\in R\,\mathrm{Im}(\varphi)$. Let $(r_n)_{n\in\mathbb{N}}\in R^{\mathbb{N}}$. Associated to this sequence we can build a module X in R-DMod as in the beginning of this section. The element $q\otimes p\otimes v_0+G\in Q\otimes_S P\otimes_R X=R(Q\otimes_S P\otimes_R X)$ because it is isomorphic to X and X satisfies this property. Therefore, we can find a $k\in\mathbb{N}$ and elements $r'_i\in R$, $q'_i\in Q$ and $p'_i\in P$ such that $q\otimes p\otimes v_0+G=\sum_i r'_i q'_i\otimes p'_i\otimes v_k+G$. Using the isomorphism Δ_X we deduce that $\varphi(q\otimes p)v_0+G=\sum_i r'_i\varphi(q'_i\otimes p'_i)v_k+G$ and then $(\varphi(q\otimes p)r_1\cdots r_k-\sum_i r'_i\varphi(q'_i\otimes p'_i))v_k+G=0$ and using the results of the beginning of this section, there exists $t\geq k$ such that $\varphi(q\otimes p)r_1\cdots r_t=\sum_i r'_i\varphi(q'_i\otimes p'_i)r_{k+1}\ldots r_t\in R\,\mathrm{Im}(\varphi)$ that is what we was looking for.

Let $(r_n)_{n\in\mathbb{N}}\in R^{\mathbb{N}}$ such that for all $n\in\mathbb{N}$, $r_1\cdots r_n\notin\mathrm{Im}(\varphi)$. Associated to this sequence, we can build the module X as in the beginning of this section. We are going to prove that the module $M=X/\mathrm{Im}(\varphi)X\in R$-DMod and Δ_M is not an isomorphism.

Let $av_k+G\in\mathrm{Im}(\varphi)X$. Then we can find a $n\geq k$, and elements $p_i\in P$ and $q_i\in Q$ such that $av_k+G=\sum_i\varphi(q_i\otimes p_i)v_n+G$, i.e., $(ar_{k+1}\cdots r_n-\sum_i\varphi(q_i\otimes p_i))v_n+G=0$. Then, there exists $m\geq n$ such that $ar_{k+1}\cdots r_m=\sum_i\varphi(q_i\otimes p_i)r_{n+1}\cdots r_m$. If we apply the previous claim, we deduce that there exists $h\geq m$ such that $ar_{k+1}\cdots r_h\in R\,\mathrm{Im}(\varphi)$. We have proved that $(av_k+G)+\mathrm{Im}(\varphi)X=0$ if and only if there exists an $h\geq k$ such that $ar_{k+1}\cdots r_h\in R\,\mathrm{Im}(\varphi)$.

Consider the canonical homomorphism $\mu : R\otimes X/\mathrm{Im}(\varphi)X\to X/\mathrm{Im}(\varphi)X$. It is clear that μ is an epimorphism because $RX=X$. Suppose $r\otimes(v_k+G)+\mathrm{Im}(\varphi)X\in\mathrm{Ker}(\mu)$, i.e $rv_k+G\in\mathrm{Im}(\varphi)X$, then there exists $h\geq k$ such that $rr_{k+1}\cdots r_h=\sum_i r'_i\varphi(q_i\otimes p_i)$

for some $r_i' \in R$, $q_i \in Q$ and $p_i \in P$, and then $r \otimes (v_k + G) + \mathrm{Im}(\varphi)X = rr_{k+1} \cdots r_h \otimes (v_h + G) + \mathrm{Im}(\varphi)X = 0$.

Module M is not 0 because $v_0 + G \notin \mathrm{Im}(\varphi)X$ $(r_1 \cdots r_n \notin \mathrm{Im}(\varphi)\ \forall n)$, but $\Delta_M = 0$, therefore Δ_M is not an isomorphism. □

As we have said in the introduction, this result lets us find a counterexample such that the categories CMod-R and DMod-R cannot be equivalent. We recall the definition of T-nilpotent (see e.g. [2, p. 314]); a subset X of a ring R is left T-nilpotent in case for any sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ there exists $n_0 \in \mathbb{N}$ such that $x_1 x_2 \cdots x_{n_0} = 0$.

Corollary 3.11. *The following conditions are equivalent*:
(1) CMod-$R = 0$.
(2) R-DMod $= 0$.
(3) *R is left T-nilpotent.*

Proof. Consider the following Morita context between R and 0, $\varphi : 0 \otimes 0 \to R$ and $\psi : 0 \otimes 0 \to 0$. This context is left acceptable if and only if R is left T-nilpotent and using the previous theorem, this is equivalent to CMod-R = CMod-0($=0$) and R-DMod = 0-DMod($=0$). □

With a version of the theorem for the right context we obtain a similar corollary for right T-nilpotence.

Corollary 3.12. *The following conditions are equivalent*:
(1) R-CMod $= 0$.
(2) DMod-$R = 0$.
(3) *R is right T-nilpotent.*

Then, if we could find a ring such that it were T-nilpotent on one side and not on the other, we would have a ring such that CMod-R and DMod-R cannot be equivalent, as one is zero and the other is not. The same would happen for R-DMod and R-CMod.

This kind of rings exist, we only have to consider the Jacobson radical of a ring that is perfect on one side and not on the other, see [2, Exercise 15.8]

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