Journal of Pure and Applied Algebra 31 (1984) 47-54
North-Holland

ON GOLDIE AND DUAL GOLDIE DIMENSIONS

P. GRZESZCZUK
Institute of Mathematics, University of Warsaw, Białystok Division, Akademicka 2, 15-267 Białystok, Poland

E.R. PUCZYŁOWSKI
Institute of Mathematics, University of Warsaw, PKIN, 00-901 Warsaw, Poland

Communicated by F. Van Oystaeyen
Received December 1982

The concept of the Goldie dimension of a module has been dualized in two different ways: by Fleury in [2] and by Varadarajan in [5]. Fleury's dualization was of the lattice nature while the character of Varadarajan's dualization was categorial. It appears, however, that in fact the second one is entirely of the lattice nature while the character of the first one is not. Modular lattices seem to be the right ground to study such questions. Namely the concept of the Goldie dimension of modules can be extended to modular lattices and the fact that the lattice dual to a modular one is modular itself, allows us to define the dual dimension of a lattice as the Goldie dimension of its dual copy. Moreover the Goldie dimension of the lattice $\mathcal{L}(M)$ of all submodules of a module M and the classical Goldie dimension of M appear to be the same. So do the dual dimension of $\mathcal{L}(M)$ and the dimension corank M defined by Varadarajan. This approach allows us to simplify proofs of all the main results of [4, 5] and gives new characterizations of the dimension corank. Besides, the constant occurring in the Kuroš–Ore Theorem [3] – called Kuroš-Ore dimension further on – as well as the spanning dimension defined by Fleury [2] prove special cases of the Goldie dimension.

As we are mainly interested in the above problems with reference to modules, the terminology we use throughout the paper is taken from the theory of modules. For the basic notations and results of the lattice theory we refer to [3].

Throughout the paper $\mathcal{L} = \langle L; \vee, \wedge \rangle$ will denote a modular lattice with 0 and 1 $(0 \neq 1)$. The dual lattice $\mathcal{L}^0 = \langle L; \wedge, \vee \rangle$ is modular as well. Thus for modular lattices the Duality Principle holds, i.e. if a statement Φ about lattices expressed in terms $\vee$ and $\wedge$ is true for all modular lattices, then the dual statement of Φ (obtained from Φ by interchanging $\vee$ and $\wedge$) is also true for all modular lattices.

Let for $a, b \in L$, $[a, b] = \{x \in L \mid a \leq x \leq b\}$. Obviously, $[a, b]$ is a sublattice of $\mathcal{L}$ and $[0, 1] = L$.

0022-4049/84/$3.00

For modular lattices the Isomorphism Theorem holds [3]:

A. *For any $a, b \in L$ the mapping $\varphi_b : x \mapsto x \wedge b$ is an isomorphism of $[a, a \vee b]$ and $[a \wedge b, b]$. The inverse isomorphism is $\psi_a : x \mapsto x \vee a$.*

A subset I of $L \setminus \{0\}$ is called join-independent iff for any finite subset X of I and $x \in I \setminus X$ we have $\bigvee X \wedge x = 0$, where $\bigvee X$ denotes the join of all elements of X. By dualization we obtain the meet-independency and respective results for this notion.

We need the following result concerning join-independency for modular lattices [3]:

B. *If I is a join-independent subset of $L \setminus \{0\}$ and x is a non-zero element of L such that for any finite $X \subseteq I$, $x \wedge \bigvee X = 0$, then the set $I \cup \{x\}$ is join-independent too.*

Zorn's lemma implies that any join-independent subset of $L \setminus \{0\}$ is contained in a maximal join-independent subset I' of $L \setminus \{0\}$. By B if $0 \neq x \in L \setminus I'$, then $\bigvee X \wedge x \neq 0$ for some finite subset X of I'.

1.

Definition 1. (a) We shall say that a non-zero element $a \in L$ is *essential* in $\mathscr{L}$ iff for any non-zero element $x \in L$, $a \wedge x \neq 0$.

(b) We shall say that a lattice $\mathscr{L}$ is *uniform* iff any non-zero element of L is essential in $\mathscr{L}$.

Remark. The terminology in Definition 1 is used as in the theory of modules. As in the lattice theory, a lattice $\mathscr{L}$ is uniform means, that 0 is a meet-irreducible element in $\mathscr{L}$.

Lemma 2. *Let $a < b < c < d$ be elements of L. If b is essential in $[a, c]$ and c is essential in $[a, d]$, then b is essential in $[a, d]$.*

Proof. Let $x \in [a, d]$ and $b \wedge x = a$. Since $b < c$, $a = (b \wedge x) \wedge c = b \wedge (x \wedge c)$. But since b is essential in $[a, c]$ and $x \wedge c \in [a, c]$, $x \wedge c = a$. Now by essentiality of c we obtain $x = a$. This means that b is essential in $[a, d]$.

The following lemma is in fact the crucial step in extending the concept of Goldie dimension of modules to modular lattices.

Lemma 3. *Let $a, b, c, d \in L$ and $b \wedge d = 0$. If a and c are essential in $[0, b]$ and $[0, d]$ respectively, then $a \vee c$ is essential in $[0, b \vee d]$.*

Proof. First we will prove that $a\vee d$ is essential in $[0,b\vee d]$. For, let x be an element of $[0,b\vee d]$ such that $x\wedge(a\vee d)=0$. This and B give $a\wedge(x\vee d)=0$ and hence $a\wedge b\wedge(x\vee d)=0$. Since a is essential in $[0,b]$ and $b\wedge(x\vee d)\in[0,b]$, $b\wedge(x\vee d)=0$. Certainly $x\wedge d=0$. Hence, using again B, we obtain $x\wedge(b\vee d)=0$. But, $x\leq b\vee d$, so $x=0$. This proves that $a\vee d$ is essential in $[0,b\vee d]$. Similarly we obtain that $a\vee c$ is essential in $[0,a\vee d]$. Now Lemma 2 completes the proof.

Using a simple induction, one can extend Lemma 3 to the following:

Corollary 4. *If $a_1,\dots,a_n,b_1,\dots,b_n$ are elements of L such that*
(i) *the set $\{b_1,\dots,b_n\}$ is join-independent,*
(ii) *a_i are essential in $[0,b_i]$ for $1\leq i\leq n$,*
then $a_1\vee\cdots\vee a_n$ is essential in $[0,b_1\vee\cdots\vee b_n]$.

Theorem 5. *The following conditions are equivalent:*
(1) *L does not contain infinite join-independent sets.*
(2) *L contains a finite join-independent set $\{a_1,\dots,a_n\}$ such that $a_1\vee\cdots\vee a_n$ is essential in $\mathscr{L}$ and the lattices $[0,a_i]$ are uniform for $1\leq i\leq n$.*
(3) $\sup\{k \mid L$ *contains a join-independent subset of cardinality equal to* $k\}=n<\infty$.
(4) *For any sequence $a_1\leq a_2\leq\cdots$ of elements of L there exists j such that for all $k\geq j$, a_j is essential in $[0,a_k]$.*

Proof. (1)$\Rightarrow$(2). Let us notice first that for any $0\neq b\in L$ there exists a non-zero element $c\leq b$ such that the lattice $[0,c]$ is uniform. If not, then by induction we will construct a sequence $c_1,c_2,\dots$ of elements of $L\setminus\{0\}$ such the set $\{c_1,c_2,\dots\}$ is join-independent and, for any k, $c_1\vee\cdots\vee c_k$ is not essential in $[0,b]$. For $k=1$ the construction is clear. Now let us assume that we have constructed $c_1,\dots,c_{k-1}$. Since $c_1\vee\cdots\vee c_{k-1}$ is not essential in $[0,b]$, there exists $0\neq d\leq b$ such that $(c_1\vee\cdots\vee c_{k-1})\wedge d=0$. By the assumption the lattice $[0,d]$ is not uniform. Hence there exist $0\neq d_1,d_2\leq d$ with $d_1\wedge d_2=0$. Put $c_k=d_1$. By B $\{c_1,\dots,c_k\}$ is join-independent and $c_1\vee\cdots\vee c_k$ is not essential in $[0,b]$ as $(c_1\vee\cdots\vee c_k)\wedge d_2=0$ and $d_2\neq 0$. Thus we have an infinite join-independent set of elements of L. This contradicts (1).

Now let X be a maximal join-independent subset of the set of all elements $x\in L$ such that the lattice $[0,x]$ is uniform. By (1) the set X is finite, say $X=\{x_1,\dots,x_n\}$. We claim that $x_1\vee\cdots\vee x_n$ is essential in $\mathscr{L}$. If not, then $(x_1\vee\cdots\vee x_n)\wedge a=0$ for some $0\neq a\in L$. By the foregoing there exists an element $0\neq c\leq a$ such that the lattice $[0,c]$ is uniform. Obviously the set $\{x_1,\dots,x_n,c\}$ is join-independent. This contradicts the maximality of X.

(2)$\Rightarrow$(3). Let us assume that L contains a join-independent set $\{b_1,\dots,b_k\}$ and $k>n$. We show by induction (changing if it is needed indexes at a_j) that
(i) for any $0\leq j\leq n$ the set $\{a_1,\dots,a_j,b_{j+1},\dots,b_k\}$ is join-independent.

For $j=0$ (i) is clear. Now let $j\geq 0$ and $c=a_1\vee\cdots\vee a_j\vee b_{j+2}\vee\cdots\vee b_k$ and consider the element $(a_1\wedge c)\vee\cdots\vee(a_n\wedge c)$. Since all lattices $[0,a_1],\ldots,[0,a_n]$ are uniform, by Corollary 4, $(a_1\wedge c)\vee\cdots\vee(a_n\wedge c)$ is essential in $[0,a_1\vee\cdots\vee a_n]$ whenever $a_s\wedge c\neq 0$ for $s=1,\ldots,n$. But by (2) $a_1\vee\cdots\vee a_n$ is essential in $\mathcal{L}$, so in this case $(a_1\wedge c)\vee\cdots\vee(a_n\wedge c)$ is essential in $\mathcal{L}$. This implies immediately that c is essential in $\mathcal{L}$ which contradicts the fact that $c\wedge b_{j+1}=0$. Hence for some $1\leq s\leq n, a_s\wedge c=0$. Putting $j+1=s$, we obtain that the set $\{a_1,\ldots,a_{j+1},b_{j+2},\ldots,b_k\}$ is join-independent. Thus (i) holds.

In particular (i) implies that the set $\{a_1,\ldots,a_n,b_{n+1},\ldots,b_k\}$ is join-independent. This is impossible as $a_1\vee\cdots\vee a_n$ is essential in $\mathcal{L}$. The proof (2)$\Rightarrow$(3) is complete.

(3)$\Rightarrow$(4). If (4) is not satisfied, then there exists a chain $0\neq a_1<a_2<\cdots$ of elements of L such that for any $j\geq 1$, a_j is not essential in some $[0,a_{k(j)}]$, with $k(j)>j$. Let $\{j_m\}$ be a sequence of indexes defined as follios: $j_1=1$, $j_m=k(j_{m-1})$. By the foregoing there exist elements $0\neq a'_{j_m}\leq a_{j_{m+1}}$ with $a_{j_m}\wedge a'_{j_m}=0$. By B the set $\{a_{j_1},a'_{j_1},\ldots,a'_{j_m},\ldots\}$ is join-independent. This contradicts (3).

(4)$\Rightarrow$(1). If (1) is not satisfied, then L contains a join-independent set $\{a_1,\ldots,a_n,\ldots\}$. Then $a_1<a_1\vee a_2<a_1\vee a_2\vee a_3<\cdots$ and for any k, $(a_1\vee\cdots\vee a_k)\wedge a_{k+1}=0$. This contradicts (4).

Now we can define the Goldie dimension of a modular lattice.

Definition 6. If $\mathcal{L}$ satisfies the equivalent conditions (1)–(4) of Theorem 5, then the Goldie dimension u-d $\mathcal{L}$ of $\mathcal{L}$ is equal to n. If $\mathcal{L}$ does not satisfy the conditions, then we put u-d $\mathcal{L}=\infty$.

By Theorem 5 we obtain the following:

Corollary 7. (a) *If* u-d $\mathcal{L}=n<\infty$ *and* $a\in L$, *then* u-d$[0,a]\leq n$ *and inequality is strict iff the element a is not essential in* $\mathcal{L}$.

(b) *If* u-d $\mathcal{L}=n<\infty$ *and the set* $\{a_1,\ldots,a_k\}$ *of elements of L is join-independent, then*

$$\text{u-d}[0,a_1\vee\cdots\vee a_k]=\sum_{j=1}^{k}\text{u-d}[0,a_j]\leq n,$$

Dualization of Definition 1 leads to the following:

Definition 8. (a) We shall say that an element $a\neq 1$ of L is *small* in $\mathcal{L}$ iff for any element $x\neq 1$ of L, $a\vee x\neq 1$.

(b) We shall say that a lattice $\mathcal{L}$ is *hollow* iff any element $x\neq 1$ of L is small in $\mathcal{L}$.

Dualizing Theorem 5 we obtain:

Theorem 9. *The following conditions are equivalent:*

(1) *L does not contain infinite meet-independent sets.*

(2) *L contains a finite meet-independent set $\{a_1, \ldots, a_n\}$ such that $a_1 \wedge \cdots \wedge a_n$ is small in $\mathscr{L}$ and $[a_i, 1]$ are hollow for $1 \leq i \leq n$.*

(3) $\sup\{k \mid$ *L contains a meet-independent subset of cardinality equal to* $k\} = n < \infty$,

(4) *For any sequence $\cdots a_n \leq a_{n-1} \leq \cdots \leq a_1$ of elements of L there exists j such that for all $k \geq j$, a_j is small in $[a_k, 1]$.*

Now we can define the dual Goldie dimension of a modular lattice.

Definition 10. If $\mathscr{L}$ satisfies the equivalent conditions (1)–(4) of Theorem 9, then the dual Goldie dimension h-d $\mathscr{L}$ of $\mathscr{L}$ is equal to n. If $\mathscr{L}$ does not satisfy the conditions, then we put h-d $\mathscr{L} = \infty$.

Obviously we have h-d $\mathscr{L} =$ u-d $\mathscr{L}^0$.

Dualizing Corollary 7 we obtain:

Corollary 11. (a) *If* h-d $\mathscr{L} = n < \infty$ *and $a \in L$, then* h-d$[a, 1] \leq n$ *and inequality is strict iff the element a is not small in $\mathscr{L}$.*

(b) *If* h-d $\mathscr{L} = k < \infty$ *and the set $\{a_1, \ldots, a_n\}$ of elements of L is meet-independent, then*

$$\text{h-d}[a_1 \wedge \cdots \wedge a_n, 1] = \sum_{j=1}^{n} \text{h-d}[a_j, 1] \leq k.$$

Since the lattice $\mathscr{L}(M)$ of all submodules of a module M is modular, we can apply the above results in this case.

Let us notice that if N is a submodule of a module M, then:

(a) N is essential (small) submodule of M iff N is an essential (small) element in the lattice $\mathscr{L}(M)$.

(b) The module M is uniform (hollow) iff the lattice $\mathscr{L}(M)$ is uniform (hollow).

Now Theorem 5 gives the following well known result.

Corollary 12. *Given a module M the following conditions are equivalent:*

(1) *M does not contain an infinite set X of non-zero submodules of M such that if $N_1, \ldots, N_k \in X$, then the sum $N_1 + \cdots + N_k$ is direct.*

(2) *M contains non-zero uniform submodules $N_1, \ldots, N_n$ such that the sum $N = N_1 + \cdots + N_n$ is direct and N is an essential submodule of M.*

(3) $\sup\{k \mid$ *M contains a direct sum of k non-zero submodules*$\} = n < \infty$,

(4) *For any sequence $N_1 \subseteq N_2 \subseteq \cdots$ of submodules of M there exists j such that N_j is an essential submodule in N_k, for $k \geq j$.*

Remark. Obviously condition (1) of the above Corollary and the fact that M contains no infinite direct sum of non-zero submodules are equivalent.

Let us notice now that if N is a submodule of a module M, then the sublattice $[N,M]$ of the lattice $\mathscr{L}(M)$ is isomorphic to $\mathscr{L}(M/N)$ in a natural way.

Now Theorem 9 applied to $\mathscr{L}(M)$ gives:

Corollary 13. *For any module M the following conditions are equivalent:*

(1) *M contains no infinite set X of proper submodules such that if $N_1,\dots,N_{k+1}\in X$, then $(N_1\cap\cdots\cap N_k)+N_{k+1}=M$.*

(2) *M contains proper submodules $N_1,\dots,N_n$ such that*

(a) *$(N_1\cap\cdots\cap N_{i-1}\cap N_{i+1}\cap\cdots\cap N_n)+N_i=M$ for $1\le i\le n$,*

(b) *$N_1\cap\cdots\cap N_n$ is a small submodule of M,*

(c) *M/N_i are hollow modules for $1\le i\le n$.*

(3) $\sup\{k\mid$ *M contains proper submodules $N_1,\dots,N_k$ such that $(N_1\cap\cdots\cap N_{i-1}\cap N_{i+1}\cap\cdots\cap N_k)+N_i=M$ for $1\le i\le k\}=n<\infty$.*

(4) *For any sequence $N_1\supseteq N_2\supseteq\cdots$ of submodules of the module M there exists j such that N_j/N_k is a small submodule of M/N_k for all $k\ge j$.*

The Chinese remainder theorem (cf. [5]) implies that if $N_1,\dots,N_n$ are submodules of a module M such that $(N_1\cap\cdots\cap N_{i-1}\cap N_{i+1}\cap\cdots\cap N_n)+N_i=M$ for $1\le i\le n$, then the natural homomorphism f of M to the product $\prod_{i=1}^n M/N_i$ is 'onto'. Obviously $\ker f=N_1\cap\cdots\cap N_n$. Conversely, if f is a homomorphism of M onto the product $\prod_{i=1}^n M_i$ of non-zero modules M_i and $N_i=\ker \pi_i\circ f$, where π_i is the natural projection of $\prod_{j=1}^n M_i$, then

(i) $\qquad (N_1\cap\cdots\cap N_{i-1}\cap N_{i+1}\cap\cdots\cap N_n)+N_i=M \quad$ for $1\le i\le n$,

(ii) $\qquad \ker f=N_1\cap\cdots\cap N_n$.

Now conditions (2) and (3) of Corollary 13 can be reformulated as follows:

(2′) There exists a homomorphism f of M onto the product of n non-zero hollow modules such that $\ker f$ is a small submodule of M.

(3′) $\sup\{k\mid M$ can be homomorphically mapped onto the product of k non-zero modules$\}=n<\infty$.

Certainly (3′) implies:

(5) An infinite set X of non-zero modules such that for any k and any $M_1,\dots,M_k\in X$, M can be homomorphically mapped onto $\prod_{i=1}^k M_i$ does not exist.

By the foregoing remarks it is clear that (5) implies the condition (1) of Corollary 13. In consequence all conditions (1)–(5) and (2′), (3′) are equivalent.

In [5] Varadarajan introduced the notion of corank of a module M, defining corank $M=n$ if M satisfies condition (3′) and corank $M=\infty$ otherwise. In [4] Sarath and Varadarajan proved, in a more complicated way equivalence of (2′) and (3′). Actually corank $M=$ h-d $\mathscr{L}(M)$, so corank is simply equal to the Goldie dimension of the dual lattice $\mathscr{L}^0(M)$ of all submodules of M.

2. Now we prove that the Kuroš–Ore dimension of modular lattices is a special case of the Goldie dimension. First we prove:

Theorem 14. *Let* $\mathscr{L}=\langle L;\wedge,\wedge\rangle$ *be a modular lattice with* 0 *and* 1. *If* $1=a_1\vee\cdots\vee a_n$ *is an irredundant representation of* 1 *as a join and the lattices* $[0,a_1],\ldots,[0,a_n]$ *are hollow, then* h-d $\mathscr{L}=n$.

Proof. We proceed by induction on n. For $n=1$ the Theorem is clear. Let $n\geq 2$ and consider the lattice $[0,\hat{a}_1]$, with $\hat{a}_1=a_2\vee\cdots\vee a_n$. Obviously the element $\hat{a}_1$ is a unit in $[0,\hat{a}_1]$ and $\hat{a}_1=a_2\vee\cdots\vee a_n$ is an irredundant representation of $\hat{a}_1$ as a join and the lattices $[0,a_i]$ are hollow for $2\leq i\leq n$. By the induction assumption we have h-d$[0,\hat{a}_1]=n-1$. Since $\hat{a}_1\vee a_1=1$, by the Isomorphism Theorem A, we obtain

$$[\hat{a}_1,1]=[\hat{a}_1,\hat{a}_1\vee a_1]\simeq[\hat{a}_1\wedge a_1,a_1]\subseteq[0,a_1].$$

This implies that the lattice $[\hat{a}_1,1]$ is hollow.

If h-d $\mathscr{L}\not\leq n$, then, by Theorem 9, L contains a sequence of elements $b_1,\ldots,b_n$ such that the set $\{\hat{a}_1,b_1,\ldots,b_n\}$ is meet-independent. Now it is easy to see that

$$\left[\bigwedge_{i=1}^{n} b_i,1\right]=\left[\bigwedge_{i=1}^{n} b_i,\hat{a}_1\vee\left(\bigwedge_{i=1}^{n} b_i\right)\right]\simeq\left[\hat{a}_1\wedge\left(\bigwedge_{i=1}^{n} b_i\right),\hat{a}_1\right]\subseteq[0,\hat{a}_1].$$

Hence h-d$[0,\hat{a}_1]\geq n$. This contradicts our assumption. Therefore h-d $\mathscr{L}\leq n$.

Now let $\bar{a}_j=a_1\wedge\cdots\wedge a_{j-1}\wedge a_{j+1}\wedge\cdots\wedge a_n$ for $1\leq j\leq n$. We have

$$1\geq\bar{a}_j\vee\left(\bigwedge_{i\neq j}\bar{a}_i\right)\geq\bar{a}_j\vee a_j=1\quad\text{for } 1\leq j\leq n.$$

Hence the set $\{\bar{a}_1,\ldots,\bar{a}_n\}$ is meet-independent and h-d $\mathscr{L}\geq n$. This proves the Theorem.

Let us observe that a lattice $\mathscr{L}$ is hollow if and only if 1 is a join-irreducible element in $\mathscr{L}$. So Theorem 14 implies:

Corollary 15 [Kuroš–Ore]. *If* $1=a_1\vee\cdots\vee a_n=b_1\vee\cdots\vee b_k$ *are irredundant representations of* 1 *as a join of join-irreducible elements* $a_1,\ldots,a_n,b_1,\ldots,b_k$, *then* $k=n$.

Now we will shortly discuss the notion of spanning dimension of modules introduced by Fleury in [2]. Trying to dualize the Goldie dimension of modules Fleury considered modules satisfying the following (in some sense dual to the condition (4) of Corollary 12) condition:

(∗) For each strictly decreasing chain $M_1\supset M_2\supset\cdots$ of submodules of M, the M_i are small in M from some point on.

He proved that if a module M satisfies (∗), then

(1) $M=N_1+\cdots+N_k$ for some hollow submodules N_i $(1\leq i\leq k)$ of M.

(2) If $M=N_1+\cdots+N_k=N'_1+\cdots+N'_m$ are two irredundant representations of M as a sum of hollow submodules, then $k=m$.

Then he defined the spanning dimension of M to be equal to $\mathrm{Sd}(M)=k$.

The condition (*) can be extended to modular lattices:

(**) For each (strictly) descreasing chain $a_1>a_2>\cdots$ of elements of L, the elements a_j are small in $\mathscr{L}$ from some point on.

Let us define $G=\{\{a_1,\dots,a_k\} \mid a_i\in L,\ \bigvee_{i=1}^k a_i=1$ and the join $\bigvee_{i=1}^k a_i$ is irredundant$\}$. We can partially-order the set G putting $\{a_1,\dots,a_k\}\geq\{b_1,\dots,b_n\}$ if every a_j for $1\leq j\leq k$ is a join of some elements b_i for $1\leq i\leq n$. Let us observe that if $\{a_1,\dots,a_k\}$ is a minimal element of G, then the lattices $[0,a_i]$ are hollow. Now the König Graph Theorem (cf. [1]) implies that if $\mathscr{L}$ satisfies (**), then $(G,\geq)$ is Artinian. In consequence if a lattice $\mathscr{L}$ satisfies (**), then for some elements $a_1,\dots,a_k\in L$, $1=a_1\vee\cdots\vee a_k$, $[0,a_i]$ are hollow for $1\leq i\leq k$. Now by Corollary 15, if $1=a_1\vee\cdots\vee a_k=a_1'\vee\cdots\vee a_n'$ are two irredundant representations of 1 such that $[0,a_i]$ $(1\leq i\leq k)$ and $[0,a_j']$ $(1\leq j\leq n)$ are hollow, then $k=n$. This extends the notion of the spanning dimension of modules to modular lattices.

Remark. Obviously the condition (**) implies condition (4) of Theorem 9 but these conditions are not equivalent (in fact even the condition (*) is not equivalent to condition (4) of Corollary 13 (see [5])). Hence the spanning dimension is not the dualization of the Goldie dimension in the lattice theoretical sense.

References

[1] C. Faith, Algebra: Rings, Modules and Categories I (Springer, Berlin, 1973).
[2] P. Fleury, A note on dualizing Goldie dimension, Canad. Math. Bull. 17 (1974) 511-517.
[3] G. Grätzer, General Lattice Theory (Akademie Verlag, Berlin, 1978).
[4] B. Sarath and K. Varadarajan, Dual Goldie dimension II, Comm. Algebra 17 (1979) 1885-1899.
[5] K. Varadarajan, Dual Goldie dimension, Comm. Algebra 7 (1979) 565-610.