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ON INJECTIVE AND QUASI-CONTINUOUS MODULES

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1. Introduction

We reprove that any injective module has a direct decomposition into a directly finite and a purely infinite part ([8], Theorem 6), and we show that this decomposition has a strong uniqueness property (Theorem 1). As a consequence of these facts, and of the cancellation property of directly finite injective modules, we derive a surprisingly powerful technical result for quasi-continuous modules: the isomorphism type of the 'internal hull' of any submodule is determined by the isomorphism type of the submodule (Theorem 4). As applications, we obtain, among others, the analogue of Theorem 1 for quasi-continuous modules (Proposition 6), the cancellation property for directly finite continuous modules (Proposition 8), and a criterion for the continuity of a direct sum of continuous modules (Theorem 13).

One would expect that our technique, which amounts to a machinery for the generalization of theorems from injective to quasi-continuous modules, should allow to develop the theory of types and dimension functions [7] for quasi-continuous, instead of non-singular injective, modules.

We list a few notational and terminological conventions: $E(M)=EM$ denotes the injective hull of the right module M. $A\subset' M$ and $B\subset^{\oplus} M$ signify that A is an essential submodule, and B a direct summand, of M. 'Summand' is synonymous with 'direct summand'. "C is subisomorphic to M" means that there exists a monomorphism from C into M.

2. A decomposition theorem for injective modules

We recall that a module is named *directly finite* if it is not isomorphic to a proper direct summand of itself, and *purely infinite* if it is isomorphic to the direct sum of two copies of itself [7].

We remind the reader that a directly finite injective module has the cancellation property ([2], Proposition 5). We also list, for future reference, Proposition 5.7 of [9]: An injective module is not directly finite, if and only if it contains an infinite direct sum of nonzero pairwise isomorphic submodules, if and only if it has a nonzero purely infinite direct summand.

With these concepts, we shall prove the following decomposition theorem for arbitrary injective modules. For non-singular injectives, it was already shown in ([7], Proposition 7.4), and the decomposition is then absolutely unique. For arbitrary injectives, part of the existence statement is contained in ([8], Theorem 6). The full theorem can be deduced from its non-singular special case by means of the functorial technique outlined in [8], but we prefer to present a (relatively) short and direct argument.

Theorem 1. *Every injective module E has a direct decomposition, $E=U\oplus V$, where U is directly finite, V is purely infinite, and U and V have no nonzero isomorphic summands (or submodules). If $E=U_1\oplus V_1=U_2\oplus V_2$ are two such decompositions, then $E=U_1\oplus V_2$ holds too, and consequently $U_1\cong U_2$ and $V_1\cong V_2$.*

We begin with an auxiliary observation.

Lemma 2. *Let A be a submodule of a module C, let EA be directly finite, and let C be subisomorphic to an injective module I. Then every monomorphism $f:A\to I$ extends to a monomorphism $C\to I$.*

Proof. The given monomorphisms $f:A\to I$ and $g:C\to I$ extend to monomorphisms $\varphi:EA\to I$ and $\gamma:EC\to I$. We have $EA\oplus X=EC$, hence $I=\gamma(EC)\oplus Y=\gamma(EA)\oplus\gamma(X)\oplus Y$, as well as $I=\varphi(EA)\oplus Z$.

As $\varphi(EA)\cong EA\cong\gamma(EA)$ is directly finite and injective, hence has the cancellation property, we conclude $Z\cong\gamma(X)\oplus Y\cong X\oplus Y$. We obtain a monomorphism $\mu:X\to Z$. Then $\varphi\oplus\mu:EC=EA\oplus X\to\varphi(EA)\oplus Z=I$ is a monomorphism, whose restriction $\varphi\oplus\mu\,|\,C$ extends f. □

Proof of Theorem 1. *Step* 1: We consider the collection of triples (V,φ',φ''), where V is a submodule of E and φ',φ'' are monomorphisms of V into itself such that $V=\varphi'(V)\oplus\varphi''(V)$. We order such triples by restriction, that is

$$(V,\varphi',\varphi'')\le(W,\psi',\psi'')\quad\text{if } V\subset W \text{ and } \varphi'=\psi'\,|\,V,\ \varphi''=\psi''\,|\,V.$$

For any chain of triples $(V_\alpha,\varphi'_\alpha,\varphi''_\alpha)$, one verifies easily that $(\bigcup V_\alpha,\bigcup\varphi'_\alpha,\bigcup\varphi''_\alpha)$ is an upper bound. Thus, Zorn's Lemma applies and produces a maximal triple (V,φ',φ''). Clearly, V is purely infinite. It is also injective, since φ',φ'' extend to isomorphisms $\phi':EV\to E(\varphi'V)$, $\phi'':EV\to E(\varphi''V)$ of the injective hulls, and therefore $(V,\varphi',\varphi'')\leq(EV,\phi',\phi'')$ hence $V=EV$ follows.

The injectivity of V implies that it is a direct summand: $E=U\oplus V$. We claim that U is directly finite.

If not, then it contains a nonzero purely infinite direct summand ([9], Proposition 5.7; stated above), that is $U^{\oplus}\supset A'\oplus A''$ with isomorphisms $\alpha': A'\oplus A''\to A'$ and $\alpha'': A'\oplus A''\to A''$. We obtain

$$(V,\varphi',\varphi'')<(V\oplus A'\oplus A'',\varphi'\oplus\alpha',\varphi''\oplus\alpha''),$$

in contradiction to the maximality of (V,φ',φ'').

(So far, we have proved directly what is already stated in ([8], Theorem 6).)

Step 2: We study now a fixed but arbitrary decomposition $E=U\oplus V$, with U directly finite and V purely infinite. The collection of all pairs (A,f), where A is a submodule of U and f is a monomorphism $A\to V$, ordered by restriction, obviously allows again the application of Zorn's Lemma, and therefore contains a maximal pair (A,f).

This module A is clearly injective, since f extends to a monomorphism $EA\to V$. Therefore we obtain a decomposition $U=U'\oplus A$, where U' is trivially directly finite. We claim that the module $V'=A\oplus V$ is isomorphic to V (and consequently is purely infinite).

Indeed, $V\cong V\oplus V$ and $V=X\oplus f(A)\cong X\oplus A$ yield $V\cong X\oplus A\oplus V$, hence $V=X_1\oplus A_1\oplus V_1$ with $X_1\cong X$, $A_1\cong A$ and $V_1\cong V$. We iterate this procedure and obtain $V=X_1\oplus A_1\oplus\cdots\oplus X_n\oplus A_n\oplus V_n$ with $X_i\cong X$, $A_i\cong A$ and $V_n\cong V$. Therefore we have $V\supset\bigoplus_{i=1}^{\infty}A_i$, and consequently

$$\begin{aligned}V&=E\left(\bigoplus_{i=1}^{\infty}A_i\right)\oplus Y=A_1\oplus E\left(\bigoplus_{i=2}^{\infty}A_i\right)\oplus Y\\&\cong A\oplus E\left(\bigoplus_{i=1}^{\infty}A_i\right)\oplus Y=A\oplus V.\end{aligned}$$

So far, we have obtained a new decomposition, $E=U'\oplus V'$, again with U' directly finite and V' purely infinite. We claim now that it enjoys the additional property that U' and V' have no nonzero isomorphic submodules.

To this end, we consider a submodule B of U' which is subisomorphic to V'. Then $A\oplus B$ is subisomorphic to V, via $B\oplus A\to V'\oplus A\cong V\oplus A\cong V$. As EA is directly finite since it is a direct summand of U, we can apply Lemma 2, with $C=B\oplus A$ and $I=V$, and we obtain a monomorphic extension $\varphi: B\oplus A\to V$ of f. Consequently, $(A,f)\leq(B\oplus A,\varphi)$ holds, and the maximality of (A,f) yields $B=0$.

Step 3: We turn now to the uniqueness statement. Thus, we are given two decompositions $E=U_1\oplus V_1=U_2\oplus V_2$ with U_i directly finite, V_i purely infinite, and no nonzero isomorphic summands between U_i and V_i. The immediate goal is to show that U_1 and V_2 have no nonzero isomorphic summands either.

We claim first that for any nonzero injective module A which is subisomorphic to both U_1 and V_2, there exists a number $n\geq 1$ such that A^n is subisomorphic to U_1 but A^{n+1} is not.

Suppose, to the contrary, that A^n is subisomorphic to U_1 for all n. By induction over n, we show $U_1 = X_n \oplus A_n \oplus \cdots \oplus A_1$ with $A_i \cong A$. For $n=1$, this is true by assumption. If it holds for n, then we have $U_1 \cong X_n \oplus A^n$; but we also have $U_1 \cong Y \oplus A^{n+1}$ by supposition. Since A, being isomorphic to a summand of U_1, is directly finite and injective hence has the cancellation property, we conclude $X_n \cong Y \oplus A$. Thus we obtain $X_n = X_{n+1} \oplus A_{n+1}$ with $A_{n+1} \cong A$, as required. We deduce that U_1 contains $\bigoplus_{i=1}^{\infty} A_i$, in contradiction to the fact that it is directly finite (cf. [9], Proposition 5.7).

If U_1 and V_2 have nonzero isomorphic summands, then by our claim we can find a nonzero injective module A, which is subisomorphic to both U_1 and V_2, but such that A^2 is not subisomorphic to U_1. We obtain $U_1 = A_1 \oplus B$ hence $E = U_1 \oplus V_1 = A_1 \oplus B \oplus V_1$, as well as a monomorphism $A^2 \to V_2^2 \cong V_2 \to E$ hence $E = A_2 \oplus A_3 \oplus C$, with $A_i \cong A$. We cancel A and deduce $B \oplus V_1 \cong A_3 \oplus C$, and consequently $B \oplus V_1 = A_4 \oplus C'$. Let π denote the projection from this module to V_1 with kernel B.

We claim that $B \cap A_4 = \ker \pi \,|\, A_4$ is essential in A_4. Indeed, let X be a submodule of A_4 with $X \cap \ker \pi \,|\, A_4 = 0$. Then X is subisomorphic to U_1 via $X \hookrightarrow A_4 \cong A \to U_1$, and to V_1 via π. Therefore we have $X=0$, by assumption on the decomposition $E = U_1 \oplus V_1$.

We deduce $A_4 \cong E(B \cap A_4) \subset^{\oplus} B$, and therefore $B = A_4 \oplus D$ with $A_5 \cong A_4 \cong A$. Consequently $U_1 = A_1 \oplus B = A_1 \oplus A_5 \oplus D$ has the submodule $A_1 \oplus A_5$ which is isomorphic to A^2, contrary to our choice of A.

We have just demonstrated that U_1 and V_2 have no nonzero isomorphic summands. It follows in particular that $U_1 \cap V_2 = 0$, since the injective hulls of $U_1 \cap V_2$ in U_1 and in V_2 are isomorphic summands. Therefore we obtain $E = U_1 \oplus V_2 \oplus F$, and we deduce $V_2 \oplus F \cong V_1$ and $U_1 \oplus F \cong U_2$. This shows that F yields isomorphic summands of U_2 and V_1, and since such cannot exist, by symmetry, we conclude that $F=0$ and $E = U_1 \oplus V_2$. □

Remarks. (1) Consider an arbitrary decomposition $E = U \oplus V$ with U directly finite and V purely infinite. It seems worthwhile to emphasize that Step 2 of our proof constructs a new decomposition $E = U' \oplus V'$ with the additional property that U' and V' have no nonzero isomorphic summands. Together with the uniqueness statement, it follows that V is determined by E up to isomorphism, and that U is determined by E up to the equivalence relation generated by isomorphism and addition of such directly finite summands of E which are absorbed by V (or by E).

One deduces easily that $E = U \oplus V$ itself possesses the additional property of no nonzero summands between U and V, if and only if U is minimal, if and only if V is maximal (in the sense that there is no other decomposition $E = U'' \oplus V''$ with U'' directly finite and V'' purely infinite, and with $U'' \subsetneq U$ respectively $V \subsetneq V''$).

In view of these 'extremal' characterizations of U and V, one wonders whether one could find an even more direct existence proof which avoids our two-step construction.

(2) The uniqueness assertion of Theorem 1 lies between absolute uniqueness and uniqueness up to isomorphism. It is closely related to a notion introduced in ([1], p. 177): there, summands A, B of a module M are called *equivalent* if they have the same direct complements in M (i.e. if $A \oplus X = M$ holds if and only if $B \oplus X = M$ does).

We consider a class $\mathfrak{K}$ of direct decompositions of M, that is a subset of the set $\mathfrak{D} = \{(A, B): A \oplus B = M\}$ of all direct decompositions of M. We call $\mathfrak{K}$ *isomorphism-closed* if $(A_1, B_1) \in \mathfrak{K}$, $(A_2, B_2) \in \mathfrak{D}$, $A_1 \cong A_2$ and $B_1 \cong B_2$ imply $(A_2, B_2) \in \mathfrak{K}$; this seems to be satisfied for all naturally arising classes $\mathfrak{K}$. We say that $\mathfrak{K}$ is *exchangeable* if $(A_1, B_1) \in \mathfrak{K}$ and $(A_2, B_2) \in \mathfrak{K}$ imply $(A_1, B_2) \in \mathfrak{K}$.

It is straightforward to verify that an isomorphism-closed class $\mathfrak{K}$ is exchangeable, if and only if $(A_1, B_1) \in \mathfrak{K}$ and $(A_2, B_2) \in \mathfrak{K}$ imply $A_1 \oplus B_2 = M$, if and only if all first (and/or second) components of members of $\mathfrak{K}$ are equivalent in the sense of [1].

An example of an isomorphism-closed and exchangeable class is provided by $\mathfrak{K} = \{(U, V): U \oplus V = M, U$ directly finite, V purely infinite, U and V have no nonzero isomorphic summands$\}$, for any injective module M (Theorem 1), or even any quasi-continuous M (Proposition 6). A second example arises from the decomposition of Proposition 9.

3. A theorem for quasi-continuous modules

A module M is quasi-continuous ([10], or π-injective [6]) if any decomposition $EM = \bigoplus E_i$ of its injective hull leads to a decomposition $M = \bigoplus (M \cap E_i)$.

We remind the reader of the following hierarchy of definitions for modules: 'injective' implies 'quasi-injective' implies 'continuous' implies 'quasi-continuous'. We also recall that any one of these four properties is inherited by direct summands (cf. [10], 219–220).

We collect a number of simple fundamental facts concerning quasi-continuity and relative injectivity:

(A) If A, B are summands of a quasi-continuous module M, and if $A \cap B = 0$, then $A \oplus B$ is also a summand of M ([10], p. 219).

(B) If $A \oplus B$ is quasi-continuous, then A and B are relatively injective with respect to each other ([10], Theorem 4.2 and [6], Proposition 1.12).

(C) If A and B are relatively injective with respect to each other, and if $EA \cong EB$, then $A \cong B$ ([6], proof of Proposition 1.11).

(D) If A_i are relatively injective with respect to B_j $(i, j = 1, 2)$, then $A_1 \oplus A_2$ is relatively injective with respect to $B_1 \oplus B_2$ ([16], Definition, Proposition 1 and Lemma).

An immediate consequence of (A)–(C) is:

(E) If A, B are summands of a quasi-continuous module M, if $A \cap B = 0$, and if $EA \cong EB$, then $A \cong B$ ([6], Proposition 1.11).

Lemma 3. *A quasi-continuous module M is purely infinite (directly finite) if and only if its injective hull EM is so.*

Proof. If M is purely infinite, $M \cong M \oplus M$, then $EM \cong EM \oplus EM$, hence EM is purely infinite.

Conversely assume that EM is purely infinite, that is $EM = E_1 \oplus E_2$ with $EM \cong E_1 \cong E_2$. By quasi-continuity we obtain $M = M \cap E_1 \oplus M \cap E_2$ and $E(M \cap E_i) = E_i$, and by (B) and (C) and $E_1 \cong E_2$ we conclude $M \cap E_1 \cong M \cap E_2$. Moreover, this result and the relative injectivity of $M \cap E_1$ and $M \cap E_2$ with respect to each other imply with (D) that M and $M \cap E_i$ are relatively injective. Then, (C) and $EM \cong E_i = E(M \cap E_i)$ yield $M \cong M \cap E_i$, and we conclude that M is purely infinite.

Turning to direct finiteness, it is clear that if M is not directly finite, hence $M \cong M \oplus X$ with $X \neq 0$, then $EM \cong EM \oplus EX$ hence EM is not directly finite.

Conversely if EM is not directly finite, then it has a nonzero purely infinite summand B ([9], Proposition 5.7), $EM = B \oplus Y$. We deduce $M = M \cap B \oplus M \cap Y$, and $M \cap B \neq 0$ is purely infinite by the first half of our consideration. We conclude that M is not directly finite. □

For any submodule A of a quasi-continuous module M, there exists a summand P of M which contains A as an essential submodule (take $P = M \cap EA$). These modules P (which are just the maximal essential extensions of A in M) are referred to in the introduction as the 'internal hulls' of A in M. Our next theorem makes a strong uniqueness assertion about P.

Theorem 4. *Let M be a quasi-continuous module, let $A_i \subset' P_i \subset^{\oplus} M$ $(i = 1, 2)$, and assume $A_1 \cong A_2$. Then $P_1 \cong P_2$.*

Proof. We put $D = A_1 \cap A_2$, and we let X_i be a complement of D in A_i. Then $D \oplus X_i \subset' A_i$, hence $E_i \oplus EX_i = EA_i = EP_i$, where E_i denotes an injective hull of D in EA_i. We note $E_1 \cong E_2$, and $X_1 \cap X_2 = 0$, the latter since $X_1 \cap X_2 \subset A_1 \cap A_2 \cap X_2 = D \cap X_2 = 0$.

Writing $M = P_i \oplus Q_i$, we obtain

$$EM = EP_i \oplus EQ_i = E_i \oplus EX_i \oplus EQ_i = U_i \oplus V_i \oplus EX_i \oplus EQ_i,$$

where $E_i = U_i \oplus V_i$ are decompositions according to Theorem 1.

We conclude that

$$M = M \cap U_i \oplus M \cap V_i \oplus M \cap EX_i \oplus M \cap EQ_i$$

holds, and we check easily that

$$P_i = M \cap U_i \oplus M \cap V_i \oplus M \cap EX_i$$

(and $Q_i = M \cap EQ_i$).

Let Y be a complement of D in M. Then $D\oplus Y\subset' M$ hence $E_i\oplus EY=EM$, and factoring out EY yields an isomorphism $\sigma: E_1\to E_2$, which is determined by $\sigma(e_1)=e_2$ if and only if $e_1-e_2\in EY$.

Quasi-continuity of M and $EM=E_i\oplus EY$ imply $M=M\cap E_i\oplus M\cap EY$, and factoring out $M\cap EY$ yields an isomorphism $\sigma': M\cap E_1\to M\cap E_2$, determined by $\sigma'(m_1)=m_2$ if and only if $m_1-m_2\in M\cap EY$, if and only if $m_1-m_2\in EY$. We conclude that σ' is the restriction of σ, that is that $\sigma(M\cap E_1)=M\cap E_2$ holds true.

From $E_i=U_i\oplus V_i$ and $\sigma(E_1)=E_2$, we obtain the two decompositions $E_2=U_2\oplus V_2=\sigma(U_1)\oplus\sigma(V_1)$ of E_2. The uniqueness part of Theorem 1 yields then $E_2=U_2\oplus\sigma(V_1)=\sigma(U_1)\oplus V_2$. For the quasi-continuous module $C=M\cap E_2$, with injective hull E_2, we deduce

$$C=C\cap U_2\oplus C\cap V_2=C\cap\sigma(U_1)\oplus C\cap\sigma(V_1)=C\cap U_2\oplus C\cap\sigma(V_1)$$
$$=C\cap\sigma(U_1)\oplus C\cap V_2,$$

and therefore

$$M\cap U_2=C\cap U_2\cong C\cap\sigma(U_1)=\sigma(M\cap E_1)\cap\sigma(U_1)=\sigma(M\cap U_1)\cong M\cap U_1,$$

and similarly $M\cap V_2\cong M\cap V_1$.

The given isomorphism $A_1\cong A_2$ yields

$$U_1\oplus V_1\oplus EX_1=EA_1\cong EA_2=U_2\oplus V_2\oplus EX_2.$$

As $U_1\cong U_2$ is directly finite, we can cancel it and obtain

$$E(M\cap V_1\oplus M\cap EX_1)=V_1\oplus EX_1\cong V_2\oplus EX_2=E(M\cap V_2\oplus M\cap EX_2).$$

We shall use (C) to conclude from this that

$$M\cap V_1\oplus M\cap EX_1\cong M\cap V_2\oplus M\cap EX_2,$$

a result which together with the already established isomorphism $M\cap U_1\cong M\cap U_2$ implies the desired conclusion

$$P_1=M\cap U_1\oplus M\cap V_1\oplus M\cap EX_1\cong M\cap U_2\oplus M\cap V_2\oplus M\cap EX_2=P_2.$$

In order to justify tnis application of (C), via (D), we have to verify that $M\cap V_1$, $M\cap EX_1$ and $M\cap V_2$, $M\cap EX_2$ are relatively injective with respect to each other. This results from Lemma 3, which yields the pure infiniteness of $M\cap V_1\cong M\cap V_2$, and from (B), as follows:

$$M\cap V_1\oplus M\cap V_2\cong(M\cap V_1)^2\cong M\cap V_1\subset^{\oplus} M$$

yields that $M\cap V_1$ and $M\cap V_2$ are relatively injective.

$$M\cap V_1\oplus M\cap EX_2\cong M\cap V_2\oplus M\cap EX_2\subset^{\oplus} M$$

yields that $M\cap V_1$ and $M\cap EX_2$ are relatively injective.

$X_1\cap X_2=0$ implies $EM=EX_1\oplus EX_2\oplus F$ hence $M=M\cap EX_1\oplus M\cap EX_2\oplus M\cap F$ and yields that $M\cap EX_1$ and $M\cap EX_2$ are relatively injective. □

Remarks. (1) Theorem 4 was known in the two special cases $A_1 = A_2$ and $A_1 \cap A_2 = 0$, where the proof is much easier ([10], (2) on p. 220 and Corollary 4.7 on p. 221). Its general validity was first suggested to us by certain results on von Neumann regular rings, in Section 14 of [9], for instance (14.26). These statements are again special cases of Theorem 4, except that they assume $\aleph_0$-continuity instead of quasi-continuity.

(2) We discuss the relationship between the isomorphisms $A_1 \cong A_2$ and $P_1 \cong P_2$ in Theorem 4, and in particular whether $P_1 \cong P_2$ can be chosen so that (I) it induces the given $A_1 \cong A_2$, or (II) it induces some (possibly different) isomorphism $A_1 \cong A_2$.

(I) holds true if M is quasi-injective. (Extend $A_1 \cong A_2$ to $EP_1 = EA_1 \cong EA_2 = EP_2$, then to $EM \to EM$, and restrict to $P_1 = M \cap EP_1$.)

(I) holds in general, in the special case $A_1 \cap A_2 = 0$ (since then $P_1 \cong P_2$ and $P_1 \oplus P_2 \subset^{\oplus} M$ imply that $P_1 \oplus P_2$ is quasi-injective).

(I) does not hold in general, even if M is continuous and if $A_1 = A_2$. (For a counterexample, take the split extension $R = F \ltimes_\sigma F$ of a field F by the bimodule ${}_\sigma F$ whose left-multiplication has been modified by a proper endomorphism σ of F. Then the right-module $M = R$ is uniform of length two, hence continuous. The endomorphisms of $A = 0 \ltimes_\sigma F$ are the natural left-multiplications by the elements of F, while the endomorphisms of A which are extendable to M are the modified left-multiplications, i.e. the natural left-multiplications by the elements of $\sigma(F)$. As $\sigma(F) \subsetneq F$ holds, not every automorphism of A can be extended to M.)

(II) holds in general, in the special case $A_1 = A_2$. (Any complement of A in M defines an isomorphism $P_1 \cong P_2$ which extends the identity on A.)

(II) fails if M is not continuous. (Consider an arbitrary submodule A_1 of M which is isomorphic to a summand A_2. Then $A_1 \subset P_1$ and $A_2 = P_2$. If $P_1 \cong P_2$ can be chosen so that it induces some isomorphism $A_1 \cong A_2$, we conclude that $A_1 = P_1$ is itself a summand. hence M is continuous.)

We do not know whether (II) holds if M is continuous.

Our next result, which is just a reformulation of Theorem 4, is a powerful strengthening of fact (E).

Corollary 5. *If A and B are summands of a quasi-continuous module, with isomorphic injective hulls, then A and B are isomorphic.*

Proof. With the given isomorphism $\varphi : EA \to EB$, we define $A_2 = \varphi(A) \cap B$ and $A_1 = \varphi^{-1}(B) \cap A$. Then φ induces an isomorphism between A_1 and A_2. Since A is essential in EA, and hence $\varphi(A)$ is essential in $\varphi(EA) = EB$, and since B is also essential in EB, we conclude that A_2 is essential in EB and hence in B. Similarly we obtain that A_1 is essential in A. Therefore Theorem 4 applies to $A_1 \subset' A \subset^{\oplus} M$ and $A_2 \subset' B \subset^{\oplus} M$ and yields $A \cong B$.

(Conversely, given the assumptions $A_i \subset' P_i \subset^{\oplus} M$ and $A_1 \cong A_2$ of Theorem 4,

$EP_1 = EA_1 \cong EA_2 = EP_2$ follows, and Corollary 5 produces the conclusion $P_1 \cong P_2$ of Theorem 4.) □

4. Applications

We shall see how the preceding results can be used to extend, with little effort, many theorems from injective to (quasi-)continuous modules.

Proposition 6. *Theorem* 1 *holds for quasi-continuous modules. Moreover, if V' is any purely infinite submodule of a quasi-continuous module M, then there exists a decomposition $M = U \oplus V$, as in Theorem* 1, *such that $V' \subset V$.*

Proof. We note first that if $M = A \oplus B$ is any decomposition of a quasi-continuous module M, such that A and B have no nonzero isomorphic summands, then they have no nonzero isomorphic submodules either. Indeed, if X and Y are isomorphic submodules of A and B respectively, then the quasi-continuity implies that X and Y are essential in summands P and Q of A and B, hence of M. By Theorem 4 we obtain $P \cong Q$, and we conclude $P = 0 = Q$ hence $X = 0 = Y$.

Next, we apply Theorem 1 to EM and obtain $EM = U \oplus V$, and consequently $M = M \cap U \oplus M \cap V$. As $U = E(M \cap U)$ and $V = E(M \cap V)$ are respectively directly finite and purely infinite, the same holds true for $M \cap U$ and $M \cap V$, by Lemma 3. Clearly $M \cap U$ and $M \cap V$ cannot have nonzero isomorphic submodules, since U and V don't.

Turning to uniqueness, let $M = U_1 \oplus V_1 = U_2 \oplus V_2$ be two decompositions, as in Theorem 1. We obtain $EM = EU_i \oplus EV_i$, with EU_i directly finite and EV_i purely infinite, by Lemma 3. If X, Y are isomorphic summands of EU_i, EV_i respectively, then $X \cap U_i$ and $X \cap V_i$ are summands of U_i and V_i respectively and hence of M. As $E(X \cap U_i) = X \cong Y = E(Y \cap V_i)$ holds, Corollary 5 implies $X \cap U_i \cong Y \cap V_i$. We conclude $X \cap U_i = 0 = U \cap V_i$, hence $X = 0 = Y$. Thus, the uniqueness statement of Theorem 1 gives $E(M) = E(U_1) \oplus E(V_2)$, and therefore $M = M \cap E(U_1) \oplus M \cap E(V_2) = U_1 \oplus V_2$.

Finally, if any purely infinite submodule V' of M is given, then $E(V')$ is a purely infinite submodules of $E(M)$, and Step 1 (restrict to modules V containing the given V'!) and Step 2 of the proof of Theorem 1 produce a decomposition $E(M) = U \oplus V$ with $E(V') \subset V$. Then, the preceding considerations yield the decomposition $M = M \cap U \oplus M \cap V$, for which $V' \subset M \cap V$ holds. □

Remark. In the context of Proposition 6, it should perhaps be pointed out that a purely infinite quasi-continuous module is automatically quasi-injective (by facts (B) and (D)).

The next two results generalize main theorems of [7] and [2] from injective to (quasi)-continuous modules.

Proposition 7. *In a quasi-continuous module, isomorphic directly finite summands have isomorphic direct complements.*

Proof. We are given $M=P_1\oplus Q_1=P_2\oplus Q_2$, with $P_1\cong P_2$ directly finite. We obtain $EM=EP_i\oplus EQ_i$, with $EP_1\cong EP_2$ directly finite, by Lemma 3. Since EP_i has the cancellation property ([2], Proposition 5), we conclude $EQ_1\cong EQ_2$. Corollary 5 yields $Q_1\cong Q_2$. □

Proposition 8. *Every directly finite continuous module has the cancellation property.*

Proof. The endomorphism ring of a continuous module M has the properties that its radical factor ring is von Neumann regular, and that idempotents can be lifted ([10], Theorem 7.1). It follows from ([19], Theorems 2 and 3) that M has the finite exchange property. This information, together with Proposition 7, implies the cancellation property, by ([5], Theorem 2). □

Examples. (1) A (non-continuous) quasi-continuous directly finite module which fails to have the cancellation property: ([18], Theorem 3) gives the well known example of a commutative domain R with a stably free projective module P which is not free, in fact with $P\oplus R\cong R^{n+1}$ but $P\not\cong R^n$. Thus R does not have the cancellation property (as module over itself). Since R is uniform, it is quasi-continuous and directly finite.

(2) A non-continuous quasi-continuous directly finite module which has the exchange property, and therefore the cancellation property: Any *local* commutative domain R which is not a field.

(3) A non-continuous quasi-continuous directly finite module which fails to have the (finite) exchange property, but still has the cancellation property: The ring $\mathbb{Z}$ of integers.

Next, we generalize ([12], Theorem 1) from continuous to quasi-continuous modules.

Proposition 9. *Every quasi-continuous module M has a decomposition, $M=P\oplus Q$, where P is essential over a direct sum $\bigoplus A_k$ of indecomposable (hence uniform) summands A_k of M, and Q has no nonzero uniform submodules. If $M=P_1\oplus Q_1=P_2\oplus Q_2$ are two such decompositions, with corresponding direct sums $\bigoplus A_k$ and $\bigoplus B_j$, then $M=P_1\oplus Q_2$ holds (hence $P_1\cong P_2$ and $Q_1\cong Q_2$), and there is a bijection $k\to j$ such that $A_k\cong B_j$.*

Proof. For a quasi-continuous module M, we know $EM=P\oplus Q$, with $\oplus A_k\subset' P$, and properties as described, from [12]. We obtain $M=M\cap P\oplus M\cap Q$. Clearly $M\cap Q$ has no nonzero uniform submodules. $M\cap P$ contains $\oplus(M\cap A_k)$; and as $M\cap A_k$ is essential in A_k, $\oplus(M\cap A_k)$ is essential in $\oplus A_k$ hence in P hence in $M\cap P$. As the A_k are summands in EM, the $M\cap A_k$ are summands in M. This proves existence.

For uniqueness, let $M=P_1\oplus Q_1=P_2\oplus Q_2$, with $\oplus A_k\subset' P_1$ and $\oplus B_j\subset' P_2$, as described above. Then $EM=EP_i\oplus EQ_i$, with $\oplus EA_k\subset' EP_1$ and $\oplus EB_j\subset' EP_2$. From the known (injective) version we obtain $EM=EP_1\oplus EQ_2$, as well as $EA_k\cong EB_j$ for a suitable bijection $k\to j$. We conclude $M=M\cap EP_1\oplus M\cap EQ_2=P_1\oplus Q_2$, and Corollary 5 yields $A_k\cong B_j$. □

We recall at this point that a quasi-continuous module is actually continuous if and only if every submodule which is isomorphic to a summand is itself a summand ([10], p. 219). The next proposition was proved by Bumby [3], for injective modules.

Proposition 10. *Mutually subisomorphic continuous modules are isomorphic.*

Proof. We consider, without loss of generality, quasi-continuous modules $N\subset M$ with a monomorphism $f: M\to N$. We obtain $E(N)\subset E(M)$, and a monomorphism $f': E(M)\to E(N)$. We conclude $E(N)\cong E(M)$, by Bumby's result ([3], Theorem).

Since we have $M\cong f(M)\subset N\subset M$, and since M is continuous, we get $f(M)\subset^{\oplus} M$ hence $f(M)\subset^{\oplus} N$. As $N\subset^{\oplus} N$ holds trivially, the quasi-continuity of N and Corollary 5 imply $M\cong f(M)\cong N$. □

Examples. Our proof has only used that one of the modules M, N is continuous, and the other quasi-continuous.

(1) Two mutually subisomorphic quasi-continuous modules which are not isomorphic: $M=I$, any non-principal ideal of a commutative domain R, and $N=R$, as R-modules.

(2) Two mutually subisomorphic modules, one injective and the other not quasi-continuous, which are not isomorphic: $M=\bigoplus_{n=0}^{\infty}\mathbb{Q}$ and $N=\mathbb{Z}\oplus\bigoplus_{n=1}^{\infty}\mathbb{Q}$, as abelian groups.

Remark. As a consequence of Proposition 10, one can show that if two arbitrary modules are subisomorphic to each other, and possess continuous hulls, then these hulls are isomorphic (cf. [13] for a discussion of continuous hulls).

5. Direct sums

We determine here when the direct sum $A\oplus B$ of two quasi-continuous, continuous or quasi-injective modules shares the respective property. Due to fact (B), it is certainly necessary that A and B are injective with respect to each other. For quasi-injectivity, fact (D) shows immediately that this necessary condition is also sufficient.

We shall prove that the very same condition is also sufficient in the other two instances. This was known only in a few special cases ([12], Theorem 2; [15], Theorem 3.4; [13], Lemma 4; [4], Theorem 3).

Lemma 11. *The following statements are equivalent for a quasi-continuous module M:*

(1) *M is continuous;*

(2) *every essential monomorphism $M \to M$ is an isomorphism;*

(3) *no summand of M is isomorphic to a proper essential submodule of itself.*

Proof. That (1) implies (2), follows immediately from the definition quoted above.

(2) implies (3): If $P \cong A \subset' P \subset^{\oplus} M$, then $M = P \oplus Q \cong A \oplus Q \subset' P \oplus Q = M$. By (2), this is an isomorphism, and $A = P$ follows.

(3) implies (1): Let A be any submodule of M which is isomorphic to a summand B of M. Then $A \subset' P \subset^{\oplus} M$ holds for some P, and $B \subset' B \subset^{\oplus} M$ is trivial. As $A \cong B$ is given, $P \cong B$ follows by Theorem 4. Thus P is isomorphic to the essential submodule A of itself, and (3) yields $A = P$. We conclude that M is continuous. □

Theorem 12. *Let $M = \bigoplus_{i \in I} A_i$, and assume $\bigoplus_{i \in I} E(A_i)$ is injective. Then M is quasi-continuous, if and only if the A_i are quasi-continuous and A_j-injective for all $j \neq i$.*

Proof. We are given that all A_i are quasi-continuous and A_j-injective for $j \neq i$. That M is quasi-continuous, will be established once we show $eM \subset M$, for every idempotent e of the endomorphism ring of $E(M)$.

As $E(M) = \bigoplus_{i \in I} E(A_i)$ holds, e can be written as a matrix $e = (\varepsilon_{ik})$, with $\varepsilon_{ik} \in \hom_R(E(A_k), E(A_i))$. The A_k-injectivity of A_i yields $\varepsilon_{ik}(A_k) \subset A_i$, for all $k \neq i$. Thus, it suffices to establish $\varepsilon_{ii}(A_i) \subset A_i$.

$e = e^2$ means $\varepsilon_{ik} = \sum_j \varepsilon_{ij}\varepsilon_{jk}$. We write $\beta_i = \sum_{j \neq i} \varepsilon_{ij}\varepsilon_{ji}$ and obtain $\varepsilon_{ii} - \varepsilon_{ii}^2 = \beta_i : E(A_i) \to E(A_i)$. With $K_i = \ker \beta_i$, we have $\varepsilon_{ii} - \varepsilon_{ii}^2 \mid K_i = 0$. Since $\beta_i \varepsilon_{ii} = (\varepsilon_{ii} - \varepsilon_{ii}^2)\varepsilon_{ii} = \varepsilon_{ii}\beta_i$ holds, we conclude $\varepsilon_{ii}(K_i) \subset K_i$. Therefore, $\varepsilon_{ii} \mid K_i$ is an idempotent in the endomorphism ring of K_i, and produces a direct decomposition $K_i = X_i \oplus Y_i$, where $X_i = \varepsilon_{ii}(K_i)$ and $Y_i = \ker \varepsilon_{ii} \cap K_i$.

$X_i \oplus Y_i = K_i \subset E(A_i)$ yields $E(A_i) = E(X_i) \oplus F_i$, with $Y_i \subset F_i$. We conclude $A_i = A_i \cap E(X_i) \oplus A_i \cap F_i$, since A_i is quasi-continuous. We claim that $\ker(\beta_i \mid E(X_i)) = X_i$ and $\ker(\beta_i \mid F_i) = Y_i$ hold.

Indeed, if $a \in \ker(\beta_i \mid E(X_i)) = \ker \beta_i \cap E(X_i)$, then $a \in \ker \beta_i = K_i = X_i \oplus Y_i$, hence $a = x + y$, and consequently $a - x = y \in E(X_i) \cap Y_i = 0$ hence $a = x \in X_i$. The converse inclusion is trivial; and a similar argument works in the second situation.

From the homomorphisms $\beta_i \mid E(X_i) : E(X_i) \to E(A_i)$ and $1 - \varepsilon_{ii} \mid E(X_i) : E(X_i) \to$

$E(A_i)$, we obtain now an induced homomorphism $\varphi: E(A_i) \to E(A_i)$ with $\varphi(\beta_i \mid E(X_i)) = 1 - \varepsilon_{ii} \mid E(X_i)$, because $\ker(\beta_i \mid E(X_i)) = X_i \subset \ker(1 - \varepsilon_{ii} \mid E(X_i))$ holds and $E(A_i)$ is injective. This implies

$$\begin{aligned}(1-\varepsilon_{ii})(A_i \cap E(X_i)) &= \varphi\beta_i(A_i \cap E(X_i)) \\ &= \sum_{j \neq i} (\varphi\varepsilon_{ij})\varepsilon_{ji}(A_i \cap E(X_i)) \\ &\subset A_i,\end{aligned}$$

since

$$(\varphi\varepsilon_{ij})\varepsilon_{ji}(A_i \cap E(X_i)) \subset (\varphi\varepsilon_{ij})\varepsilon_{ji}(A_i) \subset (\varphi\varepsilon_{ij})(A_j) \subset A_i,$$

due to the relative injectivity of A_i and A_j. We deduce $\varepsilon_{ii}(A_i \cap E(X_i)) \subset A_i$.

Similarly, working with the homomorphisms $\beta_i \mid F_i : F_i \to E(A_i)$ and $\varepsilon_{ii} \mid F_i : F_i \to E(A_i)$, we obtain $\varepsilon_{ii}(A_i \cap F_i) \subset A_i$. This establishes $\varepsilon_{ii}(A_i) \subset A_i$, and completes the proof. □

Theorem 13. *Let $M = \bigoplus_{i \in I} A_i$, and assume that $\bigoplus_{i \in I} E(A_i)$ is injective. Then M is continuous, if and only if the A_i are continuous and A_j-injective for all $j \neq i$.*

Proof. The necessity of the conditions is clear. Conversely, if they are given, then Theorem 12 shows that M is quasi-continuous. According to Lemma 11, it suffices to show that every essential monomorphism $f: M \to M$ is onto.

As $M \cong f(M) = \bigoplus_{i \in I} f(A_i) \subset M$ is true, and M is quasi-continuous, there exist summands $f(A_i) \subset' P_i \subset^{\oplus} M$. Since $A_i \subset' A_i \subset^{\oplus} M$ holds trivially, Theorem 4 implies $A_i \cong P_i$. This yields the essential monomorphisms $A_i \cong f(A_i) \subset' P_i \cong A_i$, which become isomorphisms since the A_i are continuous. We conclude $f(A_i) = P_i \subset^{\oplus} M$, and consequently $M \cap E(f(A_i)) = f(A_i)$.

$\bigoplus_{i \in I} f(A_i) = f(M) \subset M$ yields $\bigoplus_{i \in I} E(f(A_i)) \subset E(M)$. As $f(A_i) \cong A_i$ holds, we have $E(f(A_i)) \cong E(A_i)$. Thus $\bigoplus_{i \in I} E(f(A_i))$ is injective, and is therefore a summand of $E(M)$. The quasi-continuity of M implies now $f(M) = \bigoplus_{i \in I} f(A_i) = \bigoplus_{i \in I} M \cap E(f(A_i)) \subset^{\oplus} M$. But $f(M)$ was essential in M, and we obtain $f(M) = M$. □

Remarks. The extra assumption in Theorems 12 and 13, that $\bigoplus_{i \in I} E(A_i)$ is injective, is automatically satisfied if the index set I is finite, or if the ring R is right-noetherian. (This settles, in particular, the question posed at the end of [12].) In general, it is stronger than necessary, but some extra condition is needed, as the following two examples show:

(1) Any direct sum of simple modules is quasi-injective, but the direct sum of their injective hulls need not be injective. (If it always is, and if R is left-perfect, then R is right-noetherian, hence right-artinian.)

(2) Let R be a domain, and $A_i = E(R)$ $(i = 0, 1, \ldots)$. Then $\bigoplus_{i=0}^{\infty} A_i$ need not be quasi-continuous. (If it is, then $\bigoplus_{i=1}^{\infty} A_i$ is A_0-injective, hence injective as $R \subset A_0$.

Thus, $E(R)$ is Σ-injective, and R is is a right-Ore domain ([4], Corollary 4 in Section 8).)

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