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PREPROJECTIVE COMPONENTS FOR CERTAIN ALGEBRAS

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Let Λ be a finite dimensional algebra over an algebraically closed field k.

In the study of mod Λ, the category of finitely generated Λ-modules, it has been recently recognized that the Auslander–Reiten quiver Q_Λ of Λ is a very useful tool.

We recall the definition of Q_Λ: Consider a system of representatives $\{M_i\}_{i\in I}$ of the isomorphism classes of indecomposable modules in mod Λ. Then Q_Λ has as points the objects M_i, $i\in I$. We put an arrow $M_i\to M_j$ if and only if there is an irreducible map between M_i and M_j.

Now if M_i is a non-projective indecomposable Λ-module, Dtr M_i is a non-injective indecomposable Λ-module; here tr is the transpose and $DM=\mathrm{Hom}_k(M,k)$. Hence Dtr $M_i\cong M_j$ for some M_j. In this way we obtain an operator on the non-projective objects with inverse trD.

Let C be a component of the Auslander–Reiten quiver, containing projectives.

We recall that a *section* in C is a subset S of C having the following properties:

(i) if $M\in S$, then Dtr $M\notin S$.

(ii) if $M\in S$ and $M\to N$ is an irreducible map, then either $N\in S$ or Dtr $N\in S$.

In many cases there exists a section S such that C can be embedded in a good way in $\mathbb{Z}S$, where $\mathbb{Z}S$ is the translation quiver defined by C. Reidtmann (see Section 1). This is the case for instance for hereditary algebras [3]. In this paper we will consider algebras Λ such that the category $\mathscr{P}_\Lambda$ of indecomposable projectives can be embedded in some good way in $\mathbb{Z}\Gamma$ for some finite quiver Γ. In this case, the connected components $\{C_j\}$ of Q_Λ containing projectives will be embedded in $\mathbb{Z}\Gamma$ in such a way that properties of Λ are given by geometrical properties of $\mathbb{Z}\Gamma$. We will prove that these components C_j are preprojective: that is C_j does not contain oriented cycles and for any $M\in C_j$ there is some $n=n(M)$ such that $\mathrm{Dtr}^n M$ is projective.

Each member Λ of the family of algebras constructed in this paper admits a $(\mathscr{P})$-cover in the sense of [2]. F. Larrión has shown recently the converse of this result.

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1. Preliminaries

1.1. Definitions and notation (see [4]). A *translation quiver* $D=(D,\tau)$ is a quiver D together with a subset P_0 of D_0 (the set of points of D) and injection $\tau: P_0 \to D_0$ such that:

(a) D has not loops nor double arrows.

(b) $x^- = (\tau x)^+$ for each $x \in P_0$, where x^+ (resp. x^-) denote the set of points y in D_0 which are connected by an arrow $x \to y$ (resp. $y \to x$) in D.

Let kD be the path category associated to the quiver D over the algebraically closed field k. Consider $\mathscr{M}$ the ideal of kD generated by the *mesh relations* in D:

$$\mu_x = \sum_{z \in x^-} \gamma_{z,x} \cdot \gamma_{\tau x, z} \quad (x \in P_0)$$

where $\gamma_{z,x}$ is the path given by the arrow $z \to x$.

Now consider the *Riedtmann category* of D:

$$k(D) = kD/\mathscr{M}.$$

1.2 Definition. If $D=(D,\tau)$ is a translation quiver, and $\mathscr{S} \subset D_0$, then we say that $\mathscr{S}$ is a *section in D* if and only if:

(a) if $x \in \mathscr{S}$, then $\tau x \notin \mathscr{S}$.

(b) if $x \to y$ is an arrow in D with $x \in \mathscr{S}$ then $y \in \mathscr{S}$ or $\tau y \in \mathscr{S}$.

The proof of the theorems in this paper will use induction on some partially ordered set of sections.

Now fix Γ any quiver without oriented cycles and double arrows. We will be specially interested in the translation quiver $\mathbb{Z}\Gamma$ which was defined by Riedtmann (see [4]) in the following way:

$$(\mathbb{Z}\Gamma)_0 = \Gamma_0 \times \mathbb{Z}$$

and there is an arrow

$$(x,i) \to (y,j)$$

in $\mathbb{Z}\Gamma$ if and only if $i=j$ and there is an arrow $x \to y$ in Γ, or $j=i+1$ and there is an arrow $y \to x$ in Γ. The translation τ of $\mathbb{Z}\Gamma$ and its inverse $\bar{\tau}$ are given by:

$$\tau(x,i) = (x,i-1) \quad \text{and} \quad \bar{\tau}(x,i) = (x,i+1).$$

We will study Mod $k(D)$, the category of (contravariant additive) functors from the Riedtmann category $k(D)$ to the category of abelian groups, for some translation quivers D. Namely, we will be interested in $\mathbb{Z}\Gamma$ and certains translation subquivers.

It is convenient to prove next proposition in the general case after this:

1.3. Remark (Riedtmann). Whenever we have a translation quiver D and a full subquiver D' of D, there is an induced structure of translation quiver in D'. Moreover, there is a natural way to identify the categories $k(D')$ and $k(D)/I$, where I is the ideal of $k(D)$ generated by the points of Γ which are not in D'.

1.4. Proposition. *Suppose $D=(D,\tau)$ is a translation quiver. Whenever $x\in P_0$, the following sequence of functors is exact in* Mod $k(D)$:

$$(-,\tau x)\xrightarrow{\alpha}\coprod_{z\in x^-}(-,z)\xrightarrow{\beta}\operatorname{rad}(-,x)\longrightarrow 0$$

where rad$(-,?)$ *is the Jacobson radical of $k(D)$, α is given by $((-,\gamma_{\tau x,z}))_{z\in x^-}$ and β by $((-,\gamma_{z,x}))_{z\in x^-}$. If $x\notin P_0$, then β is an isomorphism.*

Proof. Assume $x\in P_0$. Clearly $\beta\alpha=0$ by definition of $\mathcal{M}$ and it is easy to see that β is epimorphism. Now take $\sigma\in\coprod(y,z_i)$ such that $\beta(\sigma)=0$. Then $\beta(\sigma)\in\mathcal{M}(y,x)$ and so $\beta(\sigma)=\sum_j u_j\mu_{y_j}v_j$ for some $v_j\in(y,\tau y_j)$, $u_j\in(y_j,x)$ and $y_j\in P_0$.

(a) If $u_j\notin\operatorname{rad}(y_j,x)$, then $y_j=x$ and $u_j=t_j1_x$ for some $t_j\in k$.

(b) if $u_j\in\operatorname{rad}(y_j,x)$, then $u_j=\beta(u_j')$ for some $u_j'\in\coprod(y_j,z_i)$.

Therefore, considering both cases, we may write:

$$\beta(\sigma)=\sum_j\beta u_j'\mu_{y_j}v_j+t\mu_x v$$

for some $v\in(y,\tau x)$ and $t\in k$.

But $(-,\mu_x)=\beta\alpha$ and then, $\beta(\sigma)=\beta(\sum u_j'\mu_{y_j}v_j+\alpha tv)$. Since kD is free, we have $\sigma=\sum u_j'\mu_{y_j}v_j+\alpha tv$. Thus, modulo $\mathcal{M}$, we have $\sigma=\alpha tv$.

The proof of the last statement is similar (case (a) can not occur). □

2. (P)-Generating systems

Observe that $\mathcal{S}_i:=\Gamma_0\times\{i\}$ is a section in $\mathbb{Z}\Gamma$ for each $i\in\mathbb{Z}$. We will say that $\mathcal{T}\subset(\mathbb{Z}\Gamma)_0$ is *flat* if $\mathcal{T}\subset\mathcal{S}_i$ for some i and $\mathcal{T}$ is connected in $\mathbb{Z}\Gamma$. An arrow $x\to y$ in $\mathbb{Z}\Gamma$ is called flat if $\{x,y\}$ is flat.

Whenever $\mathcal{T}\subset(\mathbb{Z}\Gamma)_0$, we will use the following notation:

$$O^+(\mathcal{T})=\bigcup_{n\geq 0}\tau^n(\mathcal{T}),\qquad O^-(\mathcal{T})=\bigcup_{n\geq 0}\bar{\tau}^n(\mathcal{T})$$

and

$$O(\mathcal{T})=O^+(\mathcal{T})\cup O^-(\mathcal{T}).$$

2.1. Definition. A family $U=\{\mathcal{T}_1,\dots,\mathcal{T}_s\}$ of flat sets is called (P)-*generating system* if the following axioms are satisfied:

(a) $O(\mathcal{T}_i)\cap O(\mathcal{T}_j)=\emptyset$ whenever $i\neq j$.

(b) There are not flat arrows from $O^+(\mathcal{T}_i)$ to $\mathcal{T}_j$ if $i\neq j$.

(c) $(\mathbb{Z}\Gamma)_0=\bigcup_{i=1}^s O(\mathcal{T}_i)$.

A quiver Γ has already been fixed, now we fix a (P)-generating system U in $\mathbb{Z}\Gamma$ and we denote $\mathscr{U} = \bigcup U$.

Define $B_0 = O^+(\mathscr{U})/\mathscr{U}$ and $\bar{\mathbb{Z}}\Gamma$ be the full translation subquiver of $\mathbb{Z}\Gamma$ whose points are $(\bar{\mathbb{Z}}\Gamma)_0 = (\mathbb{Z}\Gamma)_0 \backslash B_0$.

Now consider $/\!/\Gamma$, the full translation subquiver of $\bar{\mathbb{Z}}\Gamma$ having as set of points $(/\!/\Gamma)_0 = (\bar{\mathbb{Z}}\Gamma)_0 \backslash B_1$ where

$$B_1 = \{x \in (\bar{\mathbb{Z}}\Gamma)_0 \mid \operatorname{Hom}_{k(\bar{\mathbb{Z}}\Gamma)}(-, \tau^n x) \mid_{\mathscr{U}} = 0 \text{ for some } n \geq 0\}.$$

We will study algebras of the form

$$\Lambda = \Lambda(\Gamma, U) = \operatorname{End}_{\mathscr{A}(/\!/\Gamma)}\Big(\coprod_{u \in \mathscr{U}} \mathscr{U}\Big)$$

where $\mathscr{A}(/\!/\Gamma)$ denotes the additive category generated by $k(/\!/\Gamma)$.

First, we will prove the following results which are basic in the proof of next theorems.

Let $\mathscr{F}$ be the family of flat subsets $\mathscr{S}$ of $\mathbb{Z}\Gamma$ satisfying the following conditions:

(a) There are $\mathscr{S}_1, \ldots, \mathscr{S}_t \in U$ such that $\mathscr{S} = \bigcup_{i=1}^{n} \bar{\tau}^{n_i} \mathscr{S}_i$ for some $n_i \geq 0$.

(b) $\mathscr{S}$ is maximal satisfying property (a). That is if $\mathscr{S}'$ is any other flat subset satisfying (a) and $\mathscr{S} \subset \mathscr{S}'$ then $\mathscr{S} = \mathscr{S}'$.

Observe that any $x \in \bar{\mathbb{Z}}\Gamma$ is contained in some $\mathscr{S} \in \mathscr{F}$: namely, $x \in \mathscr{S}_i$ for some i and it is clear that the connected component of $\mathscr{S}_i \cap \bar{\mathbb{Z}}\Gamma$ containing x satisfies properties (a) and (b).

If $\mathscr{S}_1, \mathscr{S}_2$ are subsets of $(\mathbb{Z}\Gamma)_0$, we write $\mathscr{S}_1 \leq \mathscr{S}_2$ if $\bar{\tau}^n \mathscr{S}_1 \subset \mathscr{S}_2$ for some $n \geq 0$.

2.2. Lemma. *$\{\mathscr{S} \cap (\bar{\mathbb{Z}}\Gamma)_0 \mid \mathscr{S} \in \mathscr{F}\}$ is a family of sections in $\bar{\mathbb{Z}}\Gamma$ which is partially ordered by $\leq$ and covers $\bar{\mathbb{Z}}\Gamma$.*

Proof. We claim that for any $\mathscr{S} \in \mathscr{F}$, $\mathscr{S} \cap (\bar{\mathbb{Z}}\Gamma)_0$ is a section in $\bar{\mathbb{Z}}\Gamma$. In order to see this, take $\mathscr{S} \in \mathscr{F}$, then $\mathscr{S} = \bigcup_{i=1}^{t} \bar{\tau}^{n_i} \mathscr{S}_i$ for some $\{\mathscr{S}_1, \ldots, \mathscr{S}_t\} \subset U$.

Assume we have an arrow $x \to y$ in $\bar{\mathbb{Z}}\Gamma$ with $x \in \mathscr{S}$. Then we have $x \in \bar{\tau}^{n_i} \mathscr{S}_i$ for some i, and by definition of (P)-generating system we also have that $y \in \bar{\tau}^{n} \mathscr{S}_{i'}$ for some $n \geq 0$ and some $\mathscr{S}_{i'} \in U$.

First case: $x, y \in \mathscr{S}_j$ for some j.

We have that $\tilde{\mathscr{S}} = \mathscr{S} \cup \bar{\tau}^{n} \mathscr{S}_{i'} \subset \mathscr{S}_j$ is a flat set which clearly satisfies property (a). Therefore $\mathscr{S} = \tilde{\mathscr{S}}$ because $\mathscr{S}$ is maximal. In particular, $y \in \mathscr{S}$.

Second case: $x \in \mathscr{S}_j$ and $y \in \mathscr{S}_{j+1}$ for some j.

By definition of (P)-generating system, we must have $n > 0$. Hence $\tau y \in \bar{\mathbb{Z}}\Gamma$, and we have as before that

$$\tilde{\mathscr{S}} = \mathscr{S} \cup \bar{\tau}^{(n-1)} \mathscr{S}_{i'} \subset \mathscr{S}_j$$

is a flat set satisfying (a). Then $\mathscr{S} = \tilde{\mathscr{S}}$ and $\tau y \in \mathscr{S}$. □

2.3. Lemma. *$\{\mathscr{S} \cap (/\!/\Gamma)_0 \mid \mathscr{S} \in \mathscr{F}\}$ is a family of sections in $\bar{\mathbb{Z}}\Gamma$ which is partially ordered by $\leq$ and covers $/\!/\Gamma$.*

Proof. Observe that whenever $x \in B_1$, then $\bar{\tau}_x \in B_1$. It follows that whenever $\mathcal{S}$ is a section in $\bar{\mathscr{T}}\Gamma$, then $\mathcal{S} \cap ({/\!\!/}\Gamma)_0$ is a section in ${/\!\!/}\Gamma$. □

3. Mod $\mathscr{U}$ in terms of Mod $k({/\!\!/}\Gamma)$

We will denote also by $\mathscr{U}$ the full subcategory of $k({/\!\!/}\Gamma)$ whose objects are the points in $\mathscr{U}$.

We have the very well known pair of adjoint functors

$$\operatorname{Mod} \mathscr{U} \underset{\operatorname{Res}}{\overset{H}{\rightleftarrows}} \operatorname{Mod} k({/\!\!/}\Gamma)$$

where Res is the restriction functor and H is its right adjoint given by

$$H(N)(x) = ((-,x)\,|_{\mathscr{U}}, N)$$

for any $N \in \operatorname{Mod} \mathscr{U}$ and $x \in ({/\!\!/}\Gamma)_0$. Here we have $\operatorname{Res} H \cong \mathrm{id}$, since $\mathscr{U}$ is a full subcategory.

We will prove that

$$\operatorname{Mod} \mathscr{U} \xrightarrow{H} \{M \mid M \cong H(N) \text{ for some } N \in \operatorname{Mod} \mathscr{U}\}$$

is an equivalence of categories and we will give a more handy description of this category.

Remark 1.3 gives a precise meaning to the following statements.

3.1. Lemma. *The following sequence of functors*

$$(-,x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-,x_i)\,|_{\mathscr{U}} \to (-,\bar{\tau}x)\,|_{\mathscr{U}} \to 0$$

is exact in Mod u *whenever* $x \in ({/\!\!/}\Gamma)_0$.

Proof. First observe that $\operatorname{rad}(-,\bar{\tau}x)\,|_{\mathscr{U}} = (-,\bar{\tau}x)\,|_{\mathscr{U}}$ because $\bar{\tau}x \notin \mathscr{U}$.

Using the fact that Res is an exact functor and Proposition 1.4, it is enough to consider the case $\bar{\tau}x \notin ({/\!\!/}\Gamma)_0$.

We will denote by $(-,y)'$ the projective corresponding to y in $k(\bar{\mathscr{T}}\Gamma)$.

Since $\bar{\tau}x \in B_1$ and $x \notin B_1$, we have that $(-,\bar{\tau}x)'\,|_{\mathscr{U}} = 0$. We know from 1.4 that

$$(-,x)'\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-,x_i)'\,|_{\mathscr{U}} \to \operatorname{rad}(-,\bar{\tau}x)'\,|_{\mathscr{U}} \to 0$$

is an exact sequence. But, as before,

$$0 = (-,\bar{\tau}x)'\,|_{\mathscr{U}} = \operatorname{rad}(-,\bar{\tau}x)'\,|_{\mathscr{U}}.$$

From 1.3, we know that there is a commutative square

$$\begin{array}{ccc} (-,x)'\,|_{\mathscr{Y}} & \longrightarrow & \coprod(-,x_i)'\,|_{\mathscr{Y}} \\ \downarrow & & \downarrow \\ (-,x)\,|_{\mathscr{Y}} & \longrightarrow & \coprod(-,x_i)\,|_{\mathscr{Y}} \end{array}$$

and we get the desired result. □

3.2. Proposition. *If $M=H(N)$ for some $N\in \operatorname{Mod} \mathscr{Y}$ then for any $x\in(\mathscr{Y}\Gamma)_0$ the following sequence is exact:*

$$0\to M(\tau x)\to \coprod_{x_i\in x^+} M(x_i)\to M(x).$$

Proof. Applying the functor $(-,N)$ to the sequence appearing in last lemma, we obtain the exact sequence

$$0\to((-,\tau x)\,|_{\mathscr{Y}},N)\to \coprod_{x_i\in x^+}((-,x_i)\,|_{\mathscr{Y}},N)\to((-,x)\,|_{\mathscr{Y}},N)$$

which is precisely the sequence we wanted. □

We will denote by $\mathscr{R}(\mathscr{Y})$ the full subcategory of Mod $k(\mathscr{Y}\Gamma)$ whose objects are the functors M such that for any $x\in(\mathscr{Y}\Gamma)_0$, the sequence:

$$0\to M(\tau x)\to \coprod_{x_i\in x^+} M(x_i)\to M(x)$$

is exact.

3.3. Lemma. *If $x\in(\mathscr{Y}\Gamma)_0$ and $M\in\mathscr{R}(\mathscr{Y})$, there is a monomorphism*

$$M(x)\xrightarrow{M(\delta_{iy})}\coprod_{y\in\mathscr{Y}} n_y M(y)$$

where each $\delta_{iy}\in(y,x)$ for $1\le iy\le n_y$.

Proof. We have in $(\mathscr{Y}\Gamma)_0$ a partial order induced by the relation $x_1<x_2$ if there is an oriented path going from x_1 to x_2. We will prove the lemma by induction on this partial order.

If $x\in\mathscr{Y}$, there is nothing to prove; this holds in particular for any x minimal.

Suppose the lemma is proved for each $z<x$ and assume $x\notin\mathscr{Y}$. Let us prove the lemma for x. Since $x\notin\mathscr{Y}$, we have $\tau x\in(\mathscr{Y}\Gamma)_0$. Then

$$M(x)\to \coprod_{x_i\in \tau x^+} M(x_i)$$

is a monomorphism because $M\in\mathscr{R}(\mathscr{Y})$. For each arrow $\tau x\to x_i$ in $\mathscr{Y}\Gamma$, we have another $x_i\to x$ in $\mathscr{Y}\Gamma$ and hence we may apply our hypothesis to each one of this x_i

obtaining monomorphisms

$$M(x_i)\to \coprod_{y\in \mathscr{U}} n_y^i M(y).$$

The composition

$$M(x)\to \coprod_{x_i\in \tau x^+} M(x_i)\to \coprod_{x_i\in \tau x^+}\ \coprod_{y\in \mathscr{U}} n_y^i M(y)$$

is the morphism we were looking for. □

3.4. Remarks. For any $M\in \mathscr{R}(\mathscr{U})$ we have:

(i) If $M|_{\mathscr{U}}=0$ then $M=0$.

(ii) If $x\in({/\!\!/}\Gamma)_0$ and $0\neq m\in M(x)$, then there is some $\phi\in(y,x)$ with $y\in\mathscr{U}$ such that $M(\phi)(m)\neq 0$.

(iii) In particular, if $(-,x)|_{\mathscr{U}}=0$ for some $x\in({/\!\!/}\Gamma)_0$, then $M(x)=0$.

3.5. Proposition. *If* $x\in({/\!\!/}\Gamma)_0$, *then* $(-,x)\in\mathscr{R}(\mathscr{U})$ *if and only if for any* $z\in({/\!\!/}\Gamma)_0$ *and* $0\neq\alpha\in(z,x)$, *there exists some* $\phi\in(y,z)$ *with* $y\in\mathscr{U}$ *and* $\alpha\phi\neq 0$.

Proof. If $(-,x)\in\mathscr{R}(\mathscr{U})$, then by last remarks the other condition is satisfied.

Now suppose this last condition and let us prove that $(-,x)\in\mathscr{R}(\mathscr{U})$. By the dual proposition of 1.4, we have an exact sequence of functors:

$$(\bar{\tau}z,-)\to\coprod_{z_i\in z^+}(z_i,-)\to \operatorname{rad}(z,-)\to 0$$

for any $z\in({/\!\!/}\Gamma)_0$. Then

$$(\bar{\tau}z,-)\to\coprod_{z_i\in z^+}(z_i,-)\to(z,-)$$

is exact and in particular

$$(\bar{\tau}z,x)\to\coprod_{z_i\in z^+}(z_i,x)\to(z,x)$$

is exact.

Now we want to see that

$$0\longrightarrow(\bar{\tau}z,x)\xrightarrow{\ \delta\ }\coprod_{z_i\in z^+}(z_i,x)$$

is exact.

If $0\neq\alpha\in(\bar{\tau}z,x)$, then we know that there is some $\phi\in(y,\bar{\tau}z)$ with $y\in\mathscr{U}$ and $\alpha\phi\neq 0$. Since $z\in({/\!\!/}\Gamma)_0$, we can assume that ϕ is some path and then it must factorize through some arrow $z_i\xrightarrow{\ \gamma\ }\bar{\tau}z$. Knowing that $\alpha\phi\neq 0$, we have $(\gamma,z)(\alpha)=\alpha\gamma\neq 0$. Hence the morphism δ, which is induced by the arrows $z_i\to\bar{\tau}z$ in ${/\!\!/}\Gamma$, is a monomorphism. □

3.6. Lemma. *If* $x\in({/\!\!/}\Gamma)_0$, $\operatorname{rad}(-,x)$ *is contained in some* $M\in\mathscr{R}(\mathscr{U})$ *and* $(-,x)|_{\mathscr{U}}\neq 0$, *then* $(-,x)\in\mathscr{R}(\mathscr{U})$.

Proof. For any $z\in(\mathscr{U}\Gamma)_0$ such that $z\neq x$ the following diagram is commutative:

$$\begin{array}{ccc}(z,x)=\operatorname{rad}(z,x) & \hookrightarrow & M(z)\\ \downarrow{\scriptstyle\bar\gamma} & & \downarrow{\scriptstyle\gamma}\\ \coprod\limits_{y\in\mathscr{U}} n_y\operatorname{rad}(z,x) & \hookrightarrow & \coprod\limits_{y\in\mathscr{U}} n_y M(y)\end{array}$$

where γ is the morphism given in 3.3, hence $\bar\gamma$ is monomorphism too; and if $z=x$, we have that $(-,x)\,|_{\mathscr{U}}\neq 0$. We can see that in both cases, the condition in the last proposition is satisfied. □

3.7. Proposition. (1) *If $M,\bar M\in\mathscr{R}(\mathscr{U})$ are such that $M\,|_{\mathscr{U}}\cong\bar M\,|_{\mathscr{U}}$ then $M\cong\bar M$.*
(2) *If we have $M\overset{\phi}{\underset{\psi}{\rightrightarrows}}\bar M$ in $\mathscr{R}(\mathscr{U})$ and $\phi\,|_{\mathscr{U}}=\psi\,|_{\mathscr{U}}$, then $\phi=\psi$.*

Proof. In order to prove (1), we are going to construct an extension $M\xrightarrow{\bar\phi}\bar M$ of the given isomorphism $M\,|_{\mathscr{U}}\xrightarrow{\phi}\bar M\,|_{\mathscr{U}}$. This will be done by induction on the partial order of the family of sections which are covering $\mathscr{U}\Gamma$ (see 2.3).

If $\mathscr{S}$ is a minimal section, then $\mathscr{S}\subset\mathscr{U}$ and $\bar\phi_x=\phi_x$ is defined for any $x\in\mathscr{S}$.

Now assume we have defined $\bar\phi_z$ for any $z\in\mathscr{S}'$, $\mathscr{S}'$ section smaller than $\mathscr{S}$, in such a way that each $\bar\phi_z$ is isomorphism and whenever we have an arrow $z\xrightarrow{\alpha}z'$ with z,z' in sections smaller than $\mathscr{S}$ then

$$\begin{array}{ccc}M(z') & \xrightarrow{M(\alpha)} & M(z)\\ \downarrow{\scriptstyle\bar\phi_{z'}} & & \downarrow{\scriptstyle\bar\phi_z}\\ \bar M(z') & \xrightarrow{\bar M(\alpha)} & \bar M(z)\end{array}$$

commutes

We will define $\bar\phi_x$ for any $x\in\mathscr{S}$ in such a way that:

(*) Whenever we have an arrow $z\xrightarrow{\alpha}x$ in $\mathscr{U}\Gamma$ the following square is commutative:

$$\begin{array}{ccc}M(x) & \xrightarrow{M(\alpha)} & M(z)\\ \downarrow{\scriptstyle\bar\phi_x} & & \downarrow{\scriptstyle\bar\phi_z}\\ \bar M(x) & \xrightarrow{\bar M(\alpha)} & \bar M(z)\end{array}$$

and this will finish the proof. This will be done by induction on the partial order of $\mathscr{S}$.

We omit the construction when x is a minimal point of $\mathscr{S}$. Assume we have defined $\bar\phi_z$ for any $z\in\mathscr{S}$ with $z<x$ and we have proved (*) for all $z<x$. We will construct $\bar\phi_x$.

If $x \in \mathscr{U}$, define $\bar{\phi}_x = \phi_x$. Let $z \xrightarrow{\alpha} x$ be an arrow with $z \in \mathscr{S}$ and $z < x$ or z in a section smaller than $\mathscr{S}$. We know by Lemma 3.3 that there is a monomorphism

$$\bar{M}(z) \xrightarrow{\beta} \coprod_{z \in \mathscr{U}} n_y \bar{M}(y).$$

Consider the diagram

$$\begin{array}{ccccc}
M(x) & \xrightarrow{M(\alpha)} & M(z) & \longrightarrow & \coprod_{y \in \mathscr{U}} n_y M(y) \\
\phi_x = \bar{\phi}_x \big\downarrow & \text{(a)} & \big\downarrow \bar{\phi}_z & \text{(b)} & \big\downarrow \coprod \bar{\phi}_y = \coprod \phi_y \\
\bar{M}(x) & \xrightarrow{\bar{M}(\alpha)} & \bar{M}(z) & \xrightarrow{\beta} & \coprod_{y \in \mathscr{U}} n_y \bar{M}(y)
\end{array}$$

By the induction hypothesis, (b) is commutative and the exterior rectangle commutes because ϕ is natural. Therefore, using that β is monomorphism, the square (a) turns out be commutative.

Now consider the case $x \notin \mathscr{U}$, then $x = \bar{\tau} z$ for some $z \in (\mathbb{N}\Gamma)_0$. Since $M, \bar{M} \in \mathscr{R}(\mathscr{U})$, the sequences:

$$0 \to M(\bar{\tau} z) \to \coprod_{z_i \in z^+} M(z_i) \to M(z)$$

and the corresponding one for $\bar{M}$ are exact. Here, each $z_i < \bar{\tau} z$ and z is in a section smaller than $\mathscr{S}$. Hence by the induction hypothesis, we have isomorphisms $\bar{\phi}_{z_i}, \bar{\phi}_z$ such that the following diagram commutes; this implies the existence of $\bar{\phi}_{\bar{\tau} z}$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M(\bar{\tau} z) & \longrightarrow & \coprod M(z_i) & \longrightarrow & M(z) \\
 & & \Big\downarrow \bar{\phi}_{\bar{\tau} z} & & \Big\downarrow \coprod \bar{\phi}_{z_i} & & \Big\downarrow \bar{\phi}_z \\
0 & \longrightarrow & \bar{M}(\bar{\tau} z) & \longrightarrow & \coprod \bar{M}(z_i) & \longrightarrow & \bar{M}(z)
\end{array}$$

The commutativity of this diagram implies the statement (*). The proof (2) is similar, and we omit it. □

3.8. Theorem.

$$\mathscr{R}(\mathscr{U}) = \{ M \in \operatorname{Mod} k(\mathbb{N}\Gamma) \mid M \cong H(N) \text{ for some } N \in \operatorname{Mod} \mathscr{U} \}.$$

Proof. First consider $M = H(N)$, then by 3.2 $M \in \mathscr{R}(\mathscr{U})$. Now suppose $M \in \mathscr{R}(\mathscr{U})$ and take $\bar{M} = H(M|_{\mathscr{U}})$. Then $\bar{M} \in \mathscr{R}(\mathscr{U})$, but $\mathscr{U}$ is a full subcategory of $k(\mathbb{N}\Gamma)$ and then $\bar{M}|_{\mathscr{U}} \cong M|_{\mathscr{U}}$. The result follows from last lemma. □

3.9. Corollary. Mod $\mathscr{U} \xrightarrow{H} \mathscr{R}(\mathscr{U})$ *is an equivalence of categories.*

Proof. Since Res $H \cong$ id, H is faithfull. By the last theorem H is dense, and, by 3.7(2), H is full. □

3.10. Remark. The equivalence in last corollary restricts to an equivalence mod $\mathscr{U} \xrightarrow{H} \mathrm{r}(\mathscr{U})$ where mod $\mathscr{U}$ and $\mathrm{r}(\mathscr{U})$ are the full subcategories of Mod $\mathscr{U}$ and $\mathscr{R}(\mathscr{U})$ respectively whose objects are functors in to the category of finite dimensional k-vector spaces.

4. Almost split sequences in Mod $\mathscr{U}$

In this section, we will restrict ourselves to the study of algebras $\Lambda = \Lambda(\Gamma, U)$ satisfying the following condition:

(∗) $(-, y_i)\,|_{\mathscr{U}} \neq 0$ for any arrow $y_i \to y$ in $\mathbb{Z}\Gamma$, whenever $y \in \mathscr{U}$.

This section will be devoted to the proof of the following result:

4.1. Theorem. *If* $(-, x)\,|_{\mathscr{U}} \neq 0$, *then:*

(A) $(-, x) \in \mathrm{r}(\mathscr{U})$.

(B) $(-, x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-, x_i)\,|_{\mathscr{U}}$ *is minimal left almost split with each* $(-, x_i) \in \mathrm{r}(\mathscr{U})$.

(C) *If* $(-, x)\,|_{\mathscr{U}}$ *is not injective, then*

$$0 \to (-, x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-, x_i)\,|_{\mathscr{U}} \to (-, \tau x)\,|_{\mathscr{U}} \to 0$$

is an almost split sequence with each $(-, x_i) \in \mathrm{r}(\mathscr{U})$ *and* $(-, \tau x) \in \mathrm{r}(\mathscr{U})$.

(D) $(-, x)\,|_{\mathscr{U}}$ *is injective if and only if* $(-, \tau x)\,|_{\mathscr{U}} = 0$.

This theorem enables us to give a good geometrical description of the preprojective components of Λ.

We will need some lemmas in the proof of the theorem, which will be done by induction on the partial order of the sections covering $\mathbb{Z}\Gamma$ (see 2.3).

4.2. Lemma. *If* $(-, x) \in \mathrm{r}(\mathscr{U})$ *and statements* (B) *and* (C) *hold for this* x, *then* (D) *also holds.*

Proof. If $(-, x)\,|_{\mathscr{U}}$ is not injective, then by (C) we must have $(-, \tau x)\,|_{\mathscr{U}} \neq 0$.

Now assume $(-, x)\,|_{\mathscr{U}}$ is injective. By assumption,

$$(-, x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-, x_i)\,|_{\mathscr{U}}$$

is minimal left almost split and, in particular, it is an epimorphism. From 3.1 we get

$$(-, x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-, x_i)\,|_{\mathscr{U}} \to (-, \tau x)\,|_{\mathscr{U}} \to 0$$

exact and therefore $(-, \tau x)\,|_{\mathscr{U}} = 0$. □

4.3. Lemma. *Take* $(-, x) \in \mathrm{r}(\mathscr{U})$ *and assume that for each arrow* $x \to x_i$ *in* $\mathbb{Z}\Gamma$ *we have* $(-, x_i) \in \mathrm{r}(\mathscr{U})$. *Then if*

$$(-, x)\,|_{\mathscr{U}} \to \coprod_{x_i \in I} (-, x_i)\,|_{\mathscr{U}}$$

is minimal left almost split with $I \subset x^+$, *then* $I = x^+$.

Proof. Assume $x \xrightarrow{\gamma_{x,x_j}} x_j$ is an arrow in $/\!/\Gamma$, then $(-,x_j) \in \mathrm{r}(\mathscr{U})$. Take $\alpha = (-,\gamma_{x,x_j})$, hence

$$\alpha\,|_{\mathscr{U}} : (-,x)\,|_{\mathscr{U}} \to (-,x_j)\,|_{\mathscr{U}}.$$

The map $i : (-,x) \to \coprod_{x_i \in I} (-,x_i)$ is given by the matrix $((-,\gamma_{x,x_i}))$ and we have $i\,|_{\mathscr{U}} : (-,x)\,|_{\mathscr{U}} \to \coprod_{x_i \in I} (-,x_i)\,|_{\mathscr{U}}$. Since $i\,|_{\mathscr{U}}$ is left almost split, $\alpha\,|_{\mathscr{U}}$ can be factorized through $i\,|_{\mathscr{U}}$. Thus $\alpha\,|_{\mathscr{U}} = \sigma \circ i\,|_{\mathscr{U}}$ for some σ. Now applying H and using 3.7(2), we get $\alpha = H(\alpha\,|_{\mathscr{U}}) = H(\sigma) \circ H(i\,|_{\mathscr{U}}) = H(\sigma) \circ i$. Moreover,

$$H(\sigma) : \coprod_{x_i \in I} (-,x_i) \to (-,x_j)$$

is induced by maps $\delta_{x_i,x_j} : x_i \to x_j$. Therefore

$$\gamma_{x,x_j} = \sum_{x_i \in I} \delta_{x_i,x_j} \circ \delta_{x,x_i}$$

and this is impossible unless $x_j \in I$. □

4.4. Lemma. *Suppose* $(-,x) \in \mathrm{r}(\mathscr{U})$ *and*

$$0 \to (-,x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-,x_i)\,|_{\mathscr{U}} \to L \to 0$$

is an almost split sequence with each $(-,x_i) \in \mathrm{r}(\mathscr{U})$. *Then* $(-,\tau x) \in \mathrm{r}(\mathscr{U})$ *and*

$$0 \to (-,x)\,|_{\mathscr{U}} \to \coprod_{x_i \in x^+} (-,x_i)\,|_{\mathscr{U}} \to (-,\tau x)\,|_{\mathscr{U}} \to 0$$

is an almost split sequence.

Proof. From Lemma 3.1 we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (-,x)\,|_{\mathscr{U}} & \longrightarrow & \coprod_{x_i \in x^+} (-,x_i)\,|_{\mathscr{U}} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow \cong & & \\
 & & (-,x)\,|_{\mathscr{U}} & \longrightarrow & \coprod_{x_i \in x^+} (-,x_i)\,|_{\mathscr{U}} & \longrightarrow & (-,\tau x)\,|_{\mathscr{U}} & \longrightarrow & 0
\end{array}$$

In particular, $\tau x \in (/\!/\Gamma)_0$. Then using 1.4 and the fact that H is left exact we get the existence of a monomorphism δ in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (-,x) & \longrightarrow & \coprod_{x_i \in x^+} (-,x_i) & \longrightarrow & H(L) & & \\
 & & \| & & \| & & \uparrow \delta & & \\
 & & (-,x) & \longrightarrow & \coprod_{x_i \in x^+} (-,x_i) & \longrightarrow & \mathrm{rad}(-,\tau x) & \longrightarrow & 0
\end{array}$$

Hence, from Lemma 3.6, $(-,\tau x) \in \mathrm{r}(\mathscr{U})$. □

Proof of 4.1. In order to prove our theorem, it will be enough to prove the following:

Assume that whenever $\mathcal{S}'$ is a section smaller than $\mathcal{S}$ the theorem holds for any point in $\mathcal{S}'$. Then the theorem is true for all the points of the section $\mathcal{S}$.

We can observe that if $\mathcal{S}$ is minimal, the hypothesis hold for $\mathcal{S}$.

Step I: (A) holds for any point in $\mathcal{S}\cap\mathcal{U}$. Suppose x is minimal in $\mathcal{S}\cap\mathcal{U}$. From 1.4, $\mathrm{rad}(-,x)\cong\coprod_{x_i\in x^-}(-,z_i)$ and using the (*)-condition, we get $(-,z_i)|_{\mathcal{U}}\neq 0$ for each arrow $z_i\to x$ in $\mathcal{U}\Gamma$. Since x is minimal in $\mathcal{S}\cap\mathcal{U}$, each $z_i\notin\mathcal{U}\cap\mathcal{S}$.

If $z_i\in\mathcal{S}$, then τz is in a section smaller than $\mathcal{S}$ and by hypothesis $(-,\tau z_i)|_{\mathcal{U}}$ is not injective and $(-,z_i)\in\mathrm{r}(\mathcal{U})$. If $z_i\notin\mathcal{S}$ then, again by hypothesis, we have $(-,z_i)\in\mathrm{r}(\mathcal{U})$. The result follows from 3.6.

Now assuming that (A) holds for any $z\in\mathcal{S}\cap\mathcal{U}$ such that $z<x$ for some $x\in\mathcal{S}\cap\mathcal{U}$, we should prove (A) for this x. The proof is similar to the above part.

Step II: (A) is true for all objects in $\mathcal{S}$. Take any $x\in\mathcal{S}$, we can assume $x\notin\mathcal{U}$. Therefore $\tau x\in\mathcal{U}\Gamma$ and τx is in a section $\mathcal{S}'$ smaller than $\mathcal{S}$. It is easy to see that (C) is true for τx and therefore $(-,x)\in\mathrm{r}(\mathcal{U})$.

Step III: The theorem is true for any x minimal in $\mathcal{S}$.

Consider the minimal left almost split map for $(-,x)|_{\mathcal{U}}:(-,x)|_{\mathcal{U}}\to\coprod B_i$ with B_i indecomposables.

If B_i is not projective, then there is an irreducible map $\mathrm{Dtr}\,B_i\to(-,x)|_{\mathcal{U}}$. If $x\notin\mathcal{U}$, then $\tau x\in(\mathcal{U}\Gamma)_0$ and applying (C) to this τx we have $\mathrm{Dtr}\,B_i\cong(-,z)|_{\mathcal{U}}$ for some arrow $z\to x$ in $\mathcal{U}\Gamma$. If $x\in\mathcal{U}$, then as we had in step I, $\mathrm{rad}(-,x)\cong\coprod_{z\in x^-}(-,z)$ and hence $\mathrm{Dtr}\,B_i\cong(-,z)|_{\mathcal{U}}$ for some arrow $z\to x$. Therefore, in both cases, $z\in\mathcal{S}'$ a section smaller than $\mathcal{S}$, since x is minimal in $\mathcal{S}$, and the result holds for z. Consequently, $B_i\cong\mathrm{trD}(-,z)|_{\mathcal{U}}\cong(-,\bar{\tau}z)|_{\mathcal{U}}$ with $(-,\bar{\tau}z)\in\mathrm{r}(\mathcal{U})$.

If B_i is projective, $B_i\cong(-,y)|_{\mathcal{U}}$ for some $y\in\mathcal{U}$ and there is an arrow $x\to y$. We have $y\in\mathcal{S}$ by 2.1(b). Then by step I, $(-,y)\in\mathrm{r}(\mathcal{U})$. Now step III follows from Lemmas 4.2, 4.3 and 4.4.

Step IV: Assume the theorem holds for any $z\in\mathcal{S}$ such that $z<x$ for some x in $\mathcal{S}$. Then the result also holds for this x.

As before, we consider the minimal left almost split map for $(-,x)|_{\mathcal{U}}:(-,x)|_{\mathcal{U}}\to\coprod B_i$ with B_i indecomposables. And again, if B_i is not projective, we have $\mathrm{Dtr}\,B_i\cong(-,z)|_{\mathcal{U}}$ for some arrow $z\to x$. Then either $z\in\mathcal{S}'$ a section smaller than $\mathcal{S}$ or $z\in\mathcal{S}$. In both cases, the theorem holds for z. Thus $B_i\cong\mathrm{trD}(-,z)|_{\mathcal{U}}\cong(-,\bar{\tau}z)|_{\mathcal{U}}$ and $(-,\bar{\tau}z)\in\mathrm{r}(\mathcal{U})$.

If B_i is projective, as before $B_i\cong(-,y)|_{\mathcal{U}}$ for some $y\in\mathcal{U}$ and some arrow $x\to y$ in $\mathcal{S}$. Hence by step I, $(-,y)\in\mathrm{r}(\mathcal{U})$. Finally, step IV follows, as before, from 4.2, 4.3 and 4.4. □

4.5. Corollary. *Λ is of finite representation type if and only if there are not nonzero paths in $k(\mathcal{U}\Gamma)$ of arbitrary length. And in this case, each connected component of the Auslander–Reiten quiver of Λ has a* (P)*-cover in the sense of* [2]. □

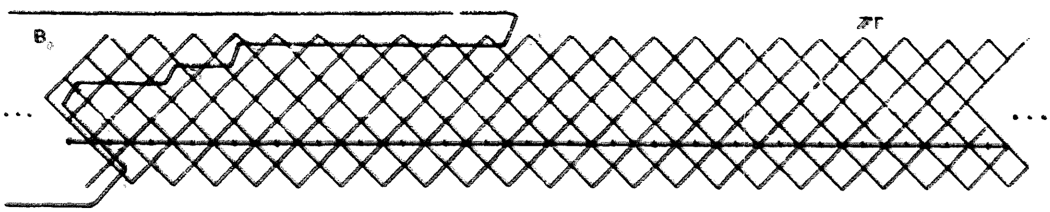


Fig. 1a.

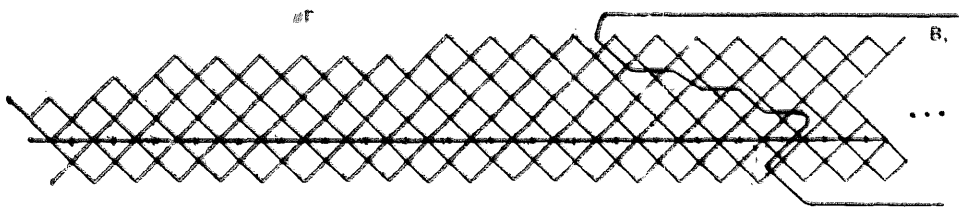


Fig. 1b.

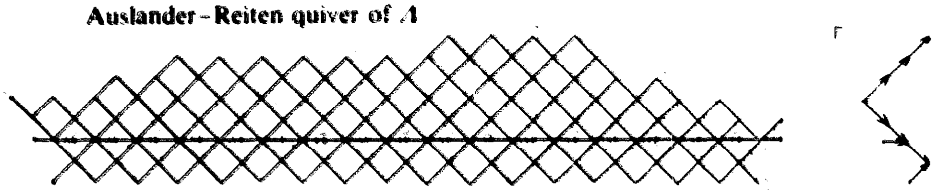
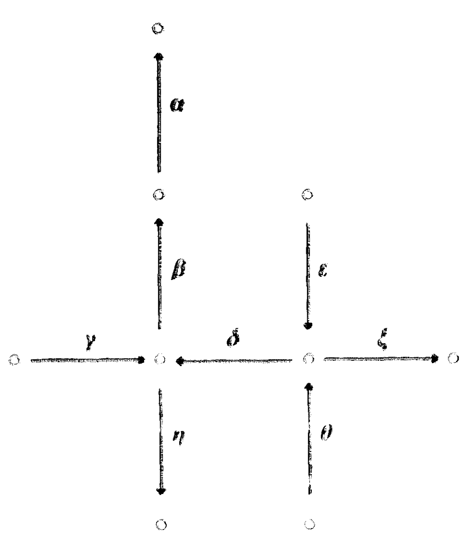


Fig. 1c.

Finally, in order to clarify underlying geometric ideas in this paper, we consider two examples. The first one (Fig. 1) corresponds to the algebra of the following bounded quiver with relations $\beta\gamma = \beta\delta\varepsilon = \xi\varepsilon = \eta\delta\theta = 0$:



(This example is considered in [3].)

The second one (Fig. 2), which is trivial in some sense, has a more revealing behavior.

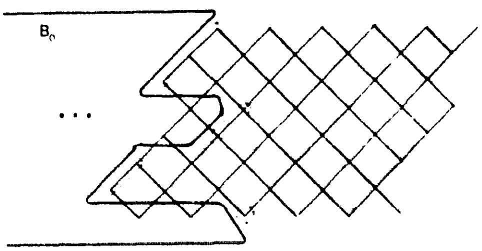


Fig. 2a.

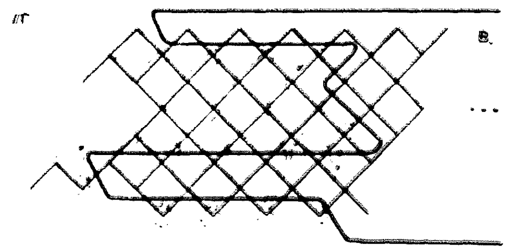


Fig. 2b.

Auslander–Reiten quiver of A

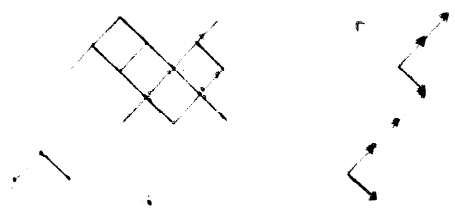


Fig. 2c.

In Figs. 1a, 2a, the sets of points enclosed in dashed lines represent the members of the (P)-generating system; in Figs. 1b, 2b, they correspond to the family of sections constructed in Section 2. Finally, in Figs. 1c, 2c, they correspond to sections of the (P)-cover in the sense of [2].

We make the convention that all arrows are directed from left to right and we omit the orientation.

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